Branching Diffusion Representations of Nonlinear PDEs

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Long Term Goal: efficient numerical methods for nonlinear, nonlocal PDEs such as

$$\max\{-\partial_t u - \mathcal{L}[u], u - \mathcal{M}[u]\} = 0 \qquad \text{on } [0,T) \times \mathcal{S}.$$

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The λ-regularized version takes the form

$$-\partial_t u - \mathcal{L}[u] + \lambda\big(u - \mathcal{M}[u]\big)_+ = 0$$

and allows us to reduce the problem to a nonlocal semilinear PDE.

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In this talk, we present an approach based on a

Branching Diffusion Representation

of the solution of the PDE.

Overview

(1) Branching Diffusion Representations

(2) Main Theoretical Results

(3) A Numerical Example from Finance

(4) Outlook, Ideas, and Conclusion

Branching Diffusion Representations

In what follows, we consider a PDE of the form

$$\begin{aligned} \partial_t u(t,x) + \mathcal{L}[u](t,x) + f(t,x,u) &= 0, && (t,x) \in [0,T] \times \mathbb{R}^d, \\ u(T,x) &= g(x), && x \in \mathbb{R}^d. \end{aligned}$$

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Here $\mathcal{L}$ is an infinitesimal generator of the form

$$
\mathcal{L}[u](t,x) = \mu(t,x)^\top \mathrm{D}_x u(t,x) + \frac{1}{2} \operatorname{tr}\big[\sigma(t,x)\sigma(t,x)^\top \mathrm{D}_x^2 u(t,x)\big]
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and the nonlocal nonlinearity is (for simplicity) of the form

$$f(t,x,u) = \sum_{i \in \mathcal{I}} c_i(t,x) \mathbb{E}\big[u(t,x+\Delta)^i\big]$$

for a random variable Δ and some $\mathcal{I} \subseteq \mathbb{N}_0$.

Towards the Feynman-Kac representation of this PDE, let

$$\mathrm{d}\overline{X}_t = \mu(t, \overline{X}_t)\mathrm{d}t + \sigma(t, \overline{X}_t)\mathrm{d}W_t, \qquad \overline{X}_0 = x.$$

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Then, at least informally,

$$u(0, x) = \mathbb{E}\Big[u\big(T, \overline{X}_T\big) + \int_0^T f\big(s, \overline{X}_s, u\big)\mathrm{d}s\Big]$$

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if W is chosen independent of Δ.

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where τ is an independent stopping time with strictly positive density ρ and

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Similarly, we can handle the sum since

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where I is an independent random variable on $\mathcal{I}$ with strictly positive probability mass function $p = (p_i)_{i\in\mathcal{I}}$.

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The last expression is the branching diffusion representation in the first generation up to time $\tau \wedge T$.

$$u(0,x) = \mathbb{E}\Big[\frac{g\big(\overline{X}_T\big)}{F(T)}\mathbb{1}_{\{\tau \geq T\}} + \frac{c_I(\tau, \overline{X}_\tau)}{\rho(\tau) p_I} u(\tau, \overline{X}_\tau + \Delta)^I \mathbb{1}_{\{\tau < T\}}\Big]$$

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For $\ell = 1, \dots, I$, let X^ℓ be an independent diffusions (driven by an independent Brownian motion W^ℓ) started in $(\tau, \overline{X}_\tau + \Delta)$. Then

$$\begin{aligned} u(0,x) &= \mathbb{E}\Big[\frac{g\big(\overline{X}_T\big)}{F(T)}\mathbb{1}_{\{\tau \geq T\}} + \frac{c_I(\tau, \overline{X}_\tau)}{\rho(\tau)p_I} u(\tau, \overline{X}_\tau + \Delta)^I \mathbb{1}_{\{\tau < T\}}\Big] \\ &= \mathbb{E}\Big[\frac{g\big(\overline{X}_T\big)}{F(T)}\mathbb{1}_{\{\tau \geq T\}} + \frac{c_I(\tau, \overline{X}_\tau)}{\rho(\tau)p_I} \mathbb{1}_{\{\tau < T\}} \prod_{\ell=1}^{I} u(\tau, X_\tau^\ell)\Big]. \end{aligned}$$

For $\ell = 1, \ldots, I$, let X^ℓ be an independent diffusions (driven by an independent Brownian motion W^ℓ) started in $(\tau, \overline{X}_\tau + \Delta)$. Then

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For each $\ell = 1, \ldots, I$, we have a another branching diffusion representation for $u(\tau, X^\ell_\tau)$ using

$$\text{independent copies } (\tau^\ell, \Delta^\ell, I^\ell, W^\ell) \qquad \text{of} \qquad (\tau, \Delta, I, W).$$

Plugging these into the expectation above yields the branching diffusion representation in the second generation up to time $(\tau + \tau^\ell) \wedge T$.

$$u(0,x) = \mathbb{E}\Big[\frac{g\big(\overline{X}_T\big)}{F(T)}\mathbb{1}_{\{\tau \geq T\}} + \frac{c_I(\tau, \overline{X}_\tau)}{\rho(\tau)p_I}\mathbb{1}_{\{\tau < T\}}\prod_{\ell=1}^{I} u(\tau, X_\tau^\ell)\Big]$$

Iterating this procedure gives the branching diffusion representation, which can be compactly written as

$$u(0,x) = \mathbb{E}[\Psi_{0,x}] \qquad \text{for} \qquad \Psi_{0,x} \triangleq \prod_{k\in\mathcal{K}} \frac{g\big(X_T^k\big)}{F(T-T^{k-})} \prod_{k\in\overline{\mathcal{K}}\setminus\mathcal{K}} \frac{c_{I^k}(T^k, X_{T^k}^k)}{\rho(T^k - T^{k-})p_{I^k}},$$

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where

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With this, we can compute $u(0,x)$ by (forward!) Monte Carlo.

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- ... and more

Main Theoretical Results

B./Hoffmann/Seifried (2021)

Let u be a classical solution of the PDE. Under some natural integrability and growth assumptions, it holds that

$$u(0,x) = \mathbb{E}[\Psi_{0,x}] \qquad \text{for} \qquad \Psi_{0,x} \triangleq \prod_{k \in \mathcal{K}} \frac{g(X_T^k)}{F(T - T^{k-})} \prod_{k \in \overline{\mathcal{K}} \setminus \mathcal{K}} \frac{c_{I^k}(T^k, X_{T^k}^k)}{\rho(T^k - T^{k-}) p_{I^k}}.$$

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Proof: Just make the arguments on the previous slides rigorous...

B./Hoffmann/Seifried (2021)

Define a function

$$u(t,x) \triangleq \mathbb{E}\big[\Psi_{t,x}\big], \qquad (t,x) \in [0,T] \times \mathbb{R}^d,$$

and, in addition to some natural integrability and growth assumptions, assume that

there exists $\varepsilon > 0$ such that $\{\Psi_{s,y} : |(s,y) - (t,x)| < \varepsilon\}$ is uniformly integrable.

Then u is a viscosity solution of the PDE.

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Then u is a viscosity solution of the PDE.

Proof: Go through the slides backwards to obtain the Feynman-Kac representation. Then use standard viscosity arguments.

B./Hoffmann/Seifried (2021)

Let $\kappa > 1$ and define

$$C_1 \triangleq \frac{\|g\|_\infty^\kappa}{F(T)^{\kappa-1}} \qquad \text{and} \qquad C_2 \triangleq \sup_{i\in\mathcal{I}, t\in[0,T]} \left(\frac{\|c_i\|_\infty}{\rho(t)p_i}\right)^{\kappa-1}$$

Then $\{\Psi_{t,x} : (t,x) \in [0,T] \times \mathbb{R}^d\}$ is bounded in $\mathrm{L}^\kappa(\mathbb{P})$, hence uniformly integrable, if either

(i) it holds that

$$\frac{C_1}{F(T)} \vee C_2 \leq 1;$$

(ii) the power series $\sum_{i\in\mathcal{I}} \|c_i\|_\infty x^i$ has infinite radius of convergence and it holds that

$$T < \int_{C_1}^{\infty} \Big(C_2 \sum_{i\in\mathcal{I}} \|c_i\|_\infty x^i\Big)^{-1} \mathrm{d}x.$$

Recall:

$$\textbf{(i)} \quad \frac{C_1}{F(T)} \vee C_2 \leq 1 \qquad \text{and} \qquad \textbf{(ii)} \quad T < \int_{C_1}^{\infty} \Big(C_2 \sum_{i \in \mathcal{I}} \|c_i\|_\infty x^i \Big)^{-1} \mathrm{d}x,$$

where

$$C_1 \triangleq \frac{\|g\|_\infty^\kappa}{F(T)^{\kappa-1}} \qquad \text{and} \qquad C_2 \triangleq \sup_{i \in \mathcal{I}, t \in [0,T]} \left(\frac{\|c_i\|_\infty}{\rho(t) p_i} \right)^{\kappa-1}.$$

Recall:

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- This shows in practice! Ideas how to mitigate this problem are very much welcome!

A Numerical Example from Finance

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- Example inspired by Pykhtin, Sokol 2013; Mercurio, Li 2015; Brigo, Pede, Petrelli 2019

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Here, τ is the default time, $h(v) = Rv_+ - v_-$ the recovery value, and V_τ is the time-τ mark-to-market value of an identical option with a systemically important counterparty which has not defaulted yet (otherwise identical and based on post-default market conditions).

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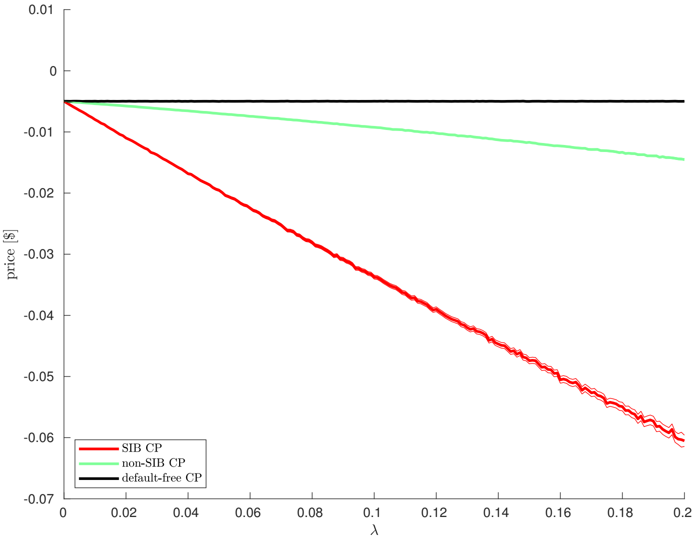
- If one approximates h by a polynomial, the associated pricing PDE can be treated with the branching diffusion approach. We consider a put on a basket with about 500 underlyings.



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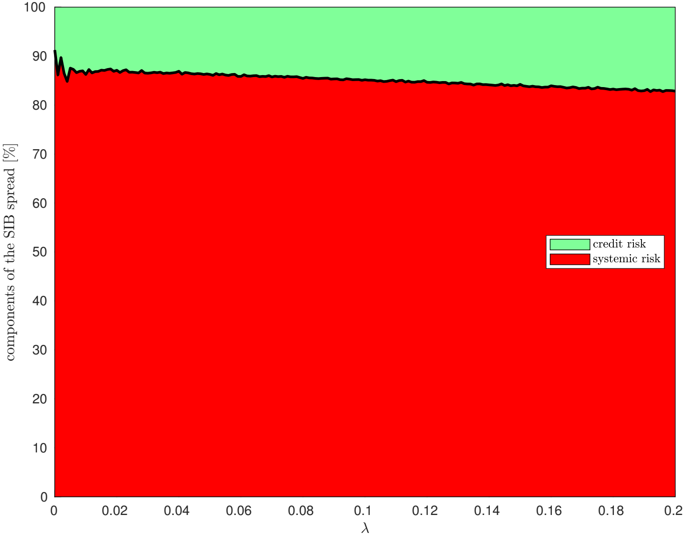
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Outlook, Ideas, and Conclusion

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Thanks for your attention!