

# Convergence of Deep Solvers for Semilinear PDEs

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- ▷ **independent** of the dimension of the state space
- ▷ but **dependent** on the dimension of the underlying Brownian motion.

## Overview

- (1) Derivation of the Deep Solver
- (2) Main Convergence Results
- (3) Some Final Remarks



## Derivation of the Deep Solver

We consider a **partial differential equation** of the form

$$\begin{aligned}u_t + \mathcal{A}[u] + f(\cdot, u, \sigma^\top D_x u) &= 0 && \text{on } [0, T) \times \mathbb{R}^d, \\u(T, \cdot) &= g && \text{on } \mathbb{R}^d,\end{aligned}$$

where  $\mathcal{A}$  is the second-order linear **differential operator** given by

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Recall that  $\mathcal{A}$  is the **infinitesimal generator** of a diffusion with dynamics

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t.$$

Suppose that  $u$  is a **classical solution** of the PDE. By Itô's formula, we obtain

$$u(t, X_t) = u(T, X_T) - \int_t^T u_t(s, X_s) + \mathcal{A}[u](s, X_s) ds - \int_t^T D_x u(s, X_s)^\top \sigma(s, X_s) dW_s$$

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Now define processes

$$Y_s \triangleq u(s, X_s) \quad \text{and} \quad Z_s \triangleq \sigma(s, X_s)^\top D_x u(s, X_s),$$

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and it follows that  $(Y, Z)$  is a solution of the **backward SDE**

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s^\top dW_s.$$

For  $I \in \mathbb{N}$ , let  $h = T/I$  denote the **step size** of the equidistant dissection

$$0 = t_0 < t_1 < \cdots < t_I = T$$

of  $[0, T]$ . Furthermore, write  $\Delta W_{i+1} = W_{t_{i+1}} - W_{t_i}$  for the Brownian increments.



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**Euler-Maruyama** discretization of the forward SDE:

$$\begin{aligned} \bar{X}_0 &\triangleq x, \\ \bar{X}_{i+1} &\triangleq \bar{X}_i + \mu(t_i, \bar{X}_i)h + \sigma(t_i, \bar{X}_i)\Delta W_{i+1}, \quad i = 0, 1, \dots, I-1. \end{aligned}$$

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**Naive Euler-Maruyama** discretization of the backward SDE:

$$\begin{aligned}\bar{Y}_I &= g(\bar{X}_I), \\ \bar{Y}_{i+1} &= \bar{Y}_i - f(t_i, \bar{X}_i, \bar{Y}_i, \bar{Z}_i)h + \bar{Z}_i^\top \Delta W_{i+1}, \quad i = 0, 1, \dots, I-1.\end{aligned}$$

**Idea:** A function  $U$  satisfies  $U \approx u$  if and only if  $U(t_i, \bar{X}_i) \approx \bar{Y}_i$  since

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$$\mathbb{L}^R(U) \triangleq \mathbb{E} \left[ |U(T, \bar{X}_T) - g(\bar{X}_T)|^2 \right] + \sum_{i=0}^{I-1} \mathbb{E} \left[ |\mathcal{L}_i^R(U)|^2 \right],$$

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**Note:** there are alternative loss functionals / approximation schemes based on similar ideas in the literature; see Beck et al. (2020) for an overview.

It turns out that Raissi's loss functional does not converge fast enough for our purposes, so we also consider the **adjusted loss functional**

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The correction term can be thought of as a consequence of a **Milstein discretization** instead of an Euler-Maruyama discretization of the backward SDE. It can be **simulated exactly**.

## Main Convergence Results

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In light of the regularity of  $u$ , we assume that the **approximating functions** are chosen from a set

$$\mathcal{U} \subset \mathcal{H}^{3,\gamma} \text{ bounded.}$$

A natural measure for the **approximation error** is

$$\mathcal{E}(U) \triangleq \max_{i=0,1,\dots,I-1} \mathbb{E} \left[ \sup_{t \in [t_i, t_{i+1}]} |U(t, X_t) - u(t, X_t)|^2 \right]$$

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Similarly, we define

$$\mathcal{D}(U) \triangleq \mathbb{E} \left[ \sum_{i=0}^{I-1} \int_{t_i}^{t_{i+1}} |\sigma(t, X_t)^\top (D_x U(t, X_t) - D_x u(t, X_t))|^2 dt \right]$$



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## Convergence Theorem I

There exists a constant  $C > 0$  which does not depend on the dimension  $d$  of the forward process  $X$  such that, for all  $h$  small enough,

$$\mathcal{E}(U) + \mathcal{D}(U) \leq Ch^{-1}\mathbb{L}(U) + Ch, \quad U \in \mathcal{U}.$$

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### Remarks:

- ▷ Heuristically, the result implies that small loss means good approximation.
- ▷ The same result is true for Raissi's loss functional  $\mathbb{L}^R(U)$ .
- ▷ The estimate is "sharp".

**Example:** Consider the PDE

$$\begin{aligned}u_t(t, x) + u_x(t, x) + 1 &= 0, & (t, x) &\in [0, T) \times \mathbb{R}, \\u(T, x) &= 1, & x &\in \mathbb{R}.\end{aligned}$$

The unique solution is given by

$$u(t, x) = T - t + 1, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

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Consider the approximating function  $U \equiv 1$ . Then

$$\mathbb{L}(U) = Ih^2 = Th$$

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**Conclusion:** The fact that the previous theorem is sharp is a consequence of

- ▷ us not imposing **additional structure** on the set  $\mathcal{U}$  of approximating functions
- ▷ and the fact that our **estimate is uniform** in  $\mathcal{U}$ .

## Convergence Theorem II

There exists a constant  $C > 0$  which does not depend on the dimension  $d$  of the forward process  $X$  such that, for all  $h$  small enough,

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- ▷ Raissi's loss functional has a lower order of convergence.

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Let  $U_h^*$  a minimizer of

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**Proof.** For  $\epsilon > 0$ , let  $U_\epsilon \in \mathcal{U}$  such that

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Now send  $\epsilon \downarrow 0$ .

**Some Final Remarks**

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