Convergence of Deep Solvers for Semilinear PDEs

Christoph Belak
Technische Universität Berlin

Joint work with Oliver Hager, Charlotte Reimers, Lotte Schnell, Frank Seifried and Maximilian Würschmidt

IFIP TC7 Conference on System Modeling and Optimization
Warsaw, July 5, 2022

Overview

(1) Derivation of the Deep Solver

(2) Main Convergence Results

(3) Some Final Remarks

Derivation of the Deep Solver

We consider a **partial differential equation** of the form

$$\begin{aligned} u_t + \mathcal{A}[u] + f(\cdot, u, \sigma^\top \mathrm{D}_x u) &= 0 && \text{on } [0,T) \times \mathbb{R}^d, \\ u(T,\cdot) &= g && \text{on } \mathbb{R}^d, \end{aligned}$$

where $\mathcal{A}$ is the second-order linear **differential operator** given by

$$\mathcal{A}[u] = \mu^\top \mathrm{D}_x u + \frac{1}{2} \operatorname{tr}\big[\sigma\sigma^\top \mathrm{D}_x^2 u\big].$$

We consider a **partial differential equation** of the form

$$\begin{aligned} u_t + \mathcal{A}[u] + f(\cdot, u, \sigma^\top \mathrm{D}_x u) &= 0 && \text{on } [0,T) \times \mathbb{R}^d, \\ u(T, \cdot) &= g && \text{on } \mathbb{R}^d, \end{aligned}$$

where $\mathcal{A}$ is the second-order linear **differential operator** given by

$$\mathcal{A}[u] = \mu^\top \mathrm{D}_x u + \frac{1}{2} \operatorname{tr}\left[\sigma\sigma^\top \mathrm{D}_x^2 u\right].$$

Recall that $\mathcal{A}$ is the **infinitesimal generator** of a diffusion with dynamics

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t.$$

We consider a **partial differential equation** of the form

$$u_t + \mathcal{A}[u] + f(\cdot, u, \sigma^\top \mathrm{D}_x u) = 0 \qquad \text{on } [0,T) \times \mathbb{R}^d,$$
$$u(T, \cdot) = g \qquad \text{on } \mathbb{R}^d,$$

where $\mathcal{A}$ is the second-order linear **differential operator** given by

$$\mathcal{A}[u] = \mu^\top \mathrm{D}_x u + \frac{1}{2} \operatorname{tr}\left[\sigma\sigma^\top \mathrm{D}_x^2 u\right].$$

Recall that $\mathcal{A}$ is the **infinitesimal generator** of a diffusion with dynamics

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t.$$

If we define processes

$$Y_t \triangleq u(t, X_t) \qquad \text{and} \qquad Z_t \triangleq \sigma(t, X_t)^\top \mathrm{D}_x u(t, X_t),$$

We consider a **partial differential equation** of the form

$$\begin{aligned} u_t + \mathcal{A}[u] + f(\cdot, u, \sigma^\top \mathrm{D}_x u) &= 0 && \text{on } [0,T) \times \mathbb{R}^d, \\ u(T,\cdot) &= g && \text{on } \mathbb{R}^d, \end{aligned}$$

where $\mathcal{A}$ is the second-order linear **differential operator** given by

$$\mathcal{A}[u] = \mu^\top \mathrm{D}_x u + \frac{1}{2}\,\mathrm{tr}\big[\sigma\sigma^\top \mathrm{D}_x^2 u\big].$$

Recall that $\mathcal{A}$ is the **infinitesimal generator** of a diffusion with dynamics

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t.$$

If we define processes

$$Y_t \triangleq u(t, X_t) \qquad \text{and} \qquad Z_t \triangleq \sigma(t, X_t)^\top \mathrm{D}_x u(t, X_t),$$

it is straightforward to see that (Y, Z) solves the **backward SDE**

$$Y_t = g(X_T) + \int_t^T f\big(s, X_s, Y_s, Z_s\big)\mathrm{d}s - \int_t^T Z_s^\top \mathrm{d}W_s.$$

For $I \in \mathbb{N}$, let $h = T/I$ denote the **step size** of the equidistant dissection

$$0 = t_0 < t_1 < \cdots < t_I = T$$

of $[0, T]$. Furthermore, write $\Delta W_{i+1} = W_{t_{i+1}} - W_{t_i}$ for the Brownian increments.

For $I \in \mathbb{N}$, let $h = T/I$ denote the **step size** of the equidistant dissection

$$0 = t_0 < t_1 < \cdots < t_I = T$$

of $[0, T]$. Furthermore, write $\Delta W_{i+1} = W_{t_{i+1}} - W_{t_i}$ for the Brownian increments.

Euler-Maruyama discretization of the forward SDE:

$$\bar{X}_0 \triangleq x, \qquad \bar{X}_{i+1} \triangleq \bar{X}_i + \mu(t_i, \bar{X}_i)h + \sigma(t_i, \bar{X}_i)\Delta W_{i+1}, \qquad i = 0, 1, \ldots, I-1.$$

For $I \in \mathbb{N}$, let $h = T/I$ denote the **step size** of the equidistant dissection

$$0 = t_0 < t_1 < \cdots < t_I = T$$

of $[0, T]$. Furthermore, write $\Delta W_{i+1} = W_{t_{i+1}} - W_{t_i}$ for the Brownian increments.

Euler-Maruyama discretization of the forward SDE:

$$\bar{X}_0 \triangleq x, \qquad \bar{X}_{i+1} \triangleq \bar{X}_i + \mu(t_i, \bar{X}_i)h + \sigma(t_i, \bar{X}_i)\Delta W_{i+1}, \qquad i = 0, 1, \ldots, I-1.$$

Naive Euler-Maruyama discretization of the backward SDE:

$$\bar{Y}_I = g(\bar{X}_I), \qquad \bar{Y}_{i+1} = \bar{Y}_i - f\big(t_i, \bar{X}_i, \bar{Y}_i, \bar{Z}_i\big)h + \bar{Z}_i^\top \Delta W_{i+1}, \qquad i = 0, 1, \ldots, I-1.$$

For $I \in \mathbb{N}$, let $h = T/I$ denote the **step size** of the equidistant dissection

$$0 = t_0 < t_1 < \cdots < t_I = T$$

of $[0, T]$. Furthermore, write $\Delta W_{i+1} = W_{t_{i+1}} - W_{t_i}$ for the Brownian increments.

Euler-Maruyama discretization of the forward SDE:

$$\bar{X}_0 \triangleq x, \qquad \bar{X}_{i+1} \triangleq \bar{X}_i + \mu(t_i, \bar{X}_i)h + \sigma(t_i, \bar{X}_i)\Delta W_{i+1}, \qquad i = 0, 1, \ldots, I-1.$$

Naive Euler-Maruyama discretization of the backward SDE:

$$\bar{Y}_I = g(\bar{X}_I), \qquad \bar{Y}_{i+1} = \bar{Y}_i - f\big(t_i, \bar{X}_i, \bar{Y}_i, \bar{Z}_i\big)h + \bar{Z}_i^\top \Delta W_{i+1}, \qquad i = 0, 1, \ldots, I-1.$$

In particular, **we expect** that

$$\begin{aligned} u(t_{i+1}, \bar{X}_{i+1}) \approx u(t_i, \bar{X}_i) - f\big(t_i, \bar{X}_i, u(t_i, \bar{X}_i), \sigma(t_i, \bar{X}_i)^\top \mathrm{D}_x u(t_i, \bar{X}_i)\big)h \\ + \mathrm{D}_x u(t_i, \bar{X}_i)^\top \sigma(t_i, \bar{X}_i)\Delta W_{i+1}. \end{aligned}$$

Idea: If we want to approximate u by a function U, one possibility is to

Minimize the error in the Euler-Maruyama discretization!

Idea: If we want to approximate u by a function U, one possibility is to

Minimize the error in the Euler-Maruyama discretization!

This leads Raissi (2018) to propose the **loss functional**

$$\mathbb{L}^{\mathrm{R}}(U) \triangleq \mathbb{E}\Big[\big|U(T,\bar{X}_I) - g(\bar{X}_I)\big|^2\Big] + \sum_{i=0}^{I-1} \mathbb{E}\Big[\big|\mathcal{L}_i^{\mathrm{R}}(U)\big|^2\Big],$$

Idea: If we want to approximate u by a function U, one possibility is to

Minimize the error in the Euler-Maruyama discretization!

This leads Raissi (2018) to propose the **loss functional**

$$\mathbb{L}^{\mathrm{R}}(U) \triangleq \mathbb{E}\Big[\big|U(T,\bar{X}_I) - g(\bar{X}_I)\big|^2\Big] + \sum_{i=0}^{I-1} \mathbb{E}\Big[\big|\mathcal{L}_i^{\mathrm{R}}(U)\big|^2\Big],$$

where

$$\begin{aligned}\mathcal{L}_i^{\mathrm{R}}(U) \triangleq U(t_{i+1},\bar{X}_{i+1}) - U(t_i,\bar{X}_i) \\ + f\big(t_i,\bar{X}_i,U(t_i,\bar{X}_i),\sigma(t_i,\bar{X}_i)^\top \mathrm{D}_x U(t_i,\bar{X}_i)\big)h \\ - \mathrm{D}_x U(t_i,\bar{X}_i)^\top \sigma(t_i,\bar{X}_i)\Delta W_{i+1}.\end{aligned}$$

Idea: If we want to approximate u by a function U, one possibility is to

Minimize the error in the Euler-Maruyama discretization!

This leads Raissi (2018) to propose the **loss functional**

$$\mathbb{L}^{\mathrm{R}}(U) \triangleq \mathbb{E}\Big[\big|U(T,\bar{X}_I) - g(\bar{X}_I)\big|^2\Big] + \sum_{i=0}^{I-1} \mathbb{E}\Big[\big|\mathcal{L}_i^{\mathrm{R}}(U)\big|^2\Big],$$

where

$$\begin{aligned}\mathcal{L}_i^{\mathrm{R}}(U) \triangleq U(t_{i+1},\bar{X}_{i+1}) - U(t_i,\bar{X}_i) \\ + f\big(t_i,\bar{X}_i,U(t_i,\bar{X}_i),\sigma(t_i,\bar{X}_i)^\top \mathrm{D}_x U(t_i,\bar{X}_i)\big)h \\ - \mathrm{D}_x U(t_i,\bar{X}_i)^\top \sigma(t_i,\bar{X}_i)\Delta W_{i+1}.\end{aligned}$$

Note: there are alternative loss functionals / approximation schemes based on similar ideas in the literature; see Beck et al. (2020) for an overview.

It turns out that Raissi's loss functional does not converge fast enough for our purposes, so we also consider the **adjusted loss functional**

$$\mathbb{L}(U) \triangleq \mathbb{E}\Big[\big|U(T, \bar{X}_I) - g(\bar{X}_I)\big|^2\Big] + \sum_{i=0}^{I-1} \mathbb{E}\Big[\big|\mathcal{L}_i(U)\big|^2\Big],$$

It turns out that Raissi's loss functional does not converge fast enough for our purposes, so we also consider the **adjusted loss functional**

$$\mathbb{L}(U) \triangleq \mathbb{E}\Big[\big|U(T,\bar{X}_I)-g(\bar{X}_I)\big|^2\Big]+\sum_{i=0}^{I-1}\mathbb{E}\Big[\big|\mathcal{L}_i(U)\big|^2\Big],$$

where

$$\begin{aligned}\mathcal{L}_i(U) \triangleq\; & U(t_{i+1},\bar{X}_{i+1})-U(t_i,\bar{X}_i)\\ & +f\big(t_i,\bar{X}_i,U(t_{i+1},\bar{X}_{i+1}),\sigma(t_i,\bar{X}_i)^\top \mathrm{D}_x U(t_i,\bar{X}_i)\big)h\\ & -\mathrm{D}_x U(t_i,\bar{X}_i)^\top\sigma(t_i,\bar{X}_i)\Delta W_{i+1}\end{aligned}$$

It turns out that Raissi's loss functional does not converge fast enough for our purposes, so we also consider the **adjusted loss functional**

$$\mathbb{L}(U) \triangleq \mathbb{E}\Big[\big|U(T,\bar{X}_I) - g(\bar{X}_I)\big|^2\Big] + \sum_{i=0}^{I-1} \mathbb{E}\Big[\big|\mathcal{L}_i(U)\big|^2\Big],$$

where

$$\begin{aligned}\mathcal{L}_i(U) \triangleq U(t_{i+1},\bar{X}_{i+1}) - U(t_i,\bar{X}_i)\\ + f\big(t_i,\bar{X}_i, U(t_{i+1},\bar{X}_{i+1}), \sigma(t_i,\bar{X}_i)^\top \mathrm{D}_x U(t_i,\bar{X}_i)\big)h\\ - \mathrm{D}_x U(t_i,\bar{X}_i)^\top \sigma(t_i,\bar{X}_i)\Delta W_{i+1}\\ - \int_{t_i}^{t_{i+1}} \Big[\mathrm{D}_x^2 U(t_i,\bar{X}_i)\big(\mu(t_i,\bar{X}_i)h + \sigma(t_i,\bar{X}_i)(W_t - W_{t_i})\big)\Big]^\top \sigma(t_i,\bar{X}_i)\mathrm{d}W_t.\end{aligned}$$

It turns out that Raissi's loss functional does not converge fast enough for our purposes, so we also consider the **adjusted loss functional**

$$\mathbb{L}(U) \triangleq \mathbb{E}\Big[\big|U(T,\bar{X}_I) - g(\bar{X}_I)\big|^2\Big] + \sum_{i=0}^{I-1} \mathbb{E}\Big[\big|\mathcal{L}_i(U)\big|^2\Big],$$

where

$$\begin{aligned}\mathcal{L}_i(U) \triangleq U(t_{i+1},\bar{X}_{i+1}) - U(t_i,\bar{X}_i) \\ + f\big(t_i,\bar{X}_i, U(t_{i+1},\bar{X}_{i+1}), \sigma(t_i,\bar{X}_i)^\top \mathrm{D}_x U(t_i,\bar{X}_i)\big)h \\ - \mathrm{D}_x U(t_i,\bar{X}_i)^\top \sigma(t_i,\bar{X}_i)\Delta W_{i+1} \\ - \int_{t_i}^{t_{i+1}} \Big[\mathrm{D}_x^2 U(t_i,\bar{X}_i)\big(\mu(t_i,\bar{X}_i)h + \sigma(t_i,\bar{X}_i)(W_t - W_{t_i})\big)\Big]^\top \sigma(t_i,\bar{X}_i)\mathrm{d}W_t.\end{aligned}$$

The correction term can be thought of as a consequence of a **Milstein discretization** instead of an Euler-Maruyama discretization of the backward SDE. It can be **simulated exactly**.

Main Convergence Results

We work under **standard assumptions** which guarantee that the PDE admits a unique solution u in the parabolic Hölder space $\mathcal{H}^{3,\gamma}$.

We work under **standard assumptions** which guarantee that the PDE admits a unique solution u in the parabolic Hölder space $\mathcal{H}^{3,\gamma}$. In particular,

- u, u_t, $\mathrm{D}_x u$, $\mathrm{D}_x^2 u$, $\mathrm{D}_x^3 u$, $\mathrm{D}_x u_t$ exist as continuous and bounded functions;
- u, $\mathrm{D}_x u$ are globally Lipschitz;
- $\mathrm{D}_x^2 u$ is Lipschitz in x, uniformly in t;
- u_t is $1/2$-Hölder in t, uniformly in x.

We work under **standard assumptions** which guarantee that the PDE admits a unique solution u in the parabolic Hölder space $\mathcal{H}^{3,\gamma}$. In particular,

- u, u_t, $\mathrm{D}_x u$, $\mathrm{D}_x^2 u$, $\mathrm{D}_x^3 u$, $\mathrm{D}_x u_t$ exist as continuous and bounded functions;
- u, $\mathrm{D}_x u$ are globally Lipschitz;
- $\mathrm{D}_x^2 u$ is Lipschitz in x, uniformly in t;
- u_t is $1/2$-Hölder in t, uniformly in x.

The precise assumptions can be found in the article and are derived from classical results; see Ladyzhenskaia et al. (1968).

We work under **standard assumptions** which guarantee that the PDE admits a unique solution u in the parabolic Hölder space $\mathcal{H}^{3,\gamma}$. In particular,

- u, u_t, $\mathrm{D}_x u$, $\mathrm{D}_x^2 u$, $\mathrm{D}_x^3 u$, $\mathrm{D}_x u_t$ exist as continuous and bounded functions;
- u, $\mathrm{D}_x u$ are globally Lipschitz;
- $\mathrm{D}_x^2 u$ is Lipschitz in x, uniformly in t;
- u_t is $1/2$-Hölder in t, uniformly in x.

The precise assumptions can be found in the article and are derived from classical results; see Ladyzhenskaia et al. (1968).

In light of the regularity of u, we assume that the **approximating functions** are chosen from a set

$$\mathcal{U} \subset \mathcal{H}^{3,\gamma} \text{ bounded.}$$

We work under **standard assumptions** which guarantee that the PDE admits a unique solution u in the parabolic Hölder space $\mathcal{H}^{3,\gamma}$. In particular,

- $u, u_t, \mathrm{D}_x u, \mathrm{D}_x^2 u, \mathrm{D}_x^3 u, \mathrm{D}_x u_t$ exist as continuous and bounded functions;
- $u, \mathrm{D}_x u$ are globally Lipschitz;
- $\mathrm{D}_x^2 u$ is Lipschitz in x, uniformly in t;
- u_t is $1/2$-Hölder in t, uniformly in x.

The precise assumptions can be found in the article and are derived from classical results; see Ladyzhenskaia et al. (1968).

In light of the regularity of u, we assume that the **approximating functions** are chosen from a set

$$\mathcal{U} \subset \mathcal{H}^{3,\gamma} \text{ bounded.}$$

Raissi (2018) suggests to choose a set of **feedforward neural networks**.

A natural measure of the **approximation error** is

$$\mathcal{E}(U) \triangleq \max_{i=0,1,\ldots,I-1} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} \left|U(t,X_t) - u(t,X_t)\right|^2\right]$$

A natural measure of the **approximation error** is

$$\mathcal{E}(U) \triangleq \max_{i=0,1,\ldots,I-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} \big|U(t,X_t) - u(t,X_t)\big|^2\Big]$$
$$= \max_{i=0,1,\ldots,I-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} \big|U(t,X_t) - Y_t\big|^2\Big].$$

A natural measure of the **approximation error** is

$$\begin{aligned}\mathcal{E}(U) &\triangleq \max_{i=0,1,\ldots,I-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} \big|U(t,X_t)-u(t,X_t)\big|^2\Big]\\ &= \max_{i=0,1,\ldots,I-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} \big|U(t,X_t)-Y_t\big|^2\Big].\end{aligned}$$

Similarly, we define

$$\mathcal{D}(U) \triangleq \mathbb{E}\Big[\sum_{i=0}^{I-1}\int_{t_i}^{t_{i+1}} \big|\sigma(t,X_t)^\top\big(\mathrm{D}_x U(t,X_t)-\mathrm{D}_x u(t,X_t)\big)\big|^2 \mathrm{d}t\Big]$$

A natural measure of the **approximation error** is

$$\begin{aligned}\mathcal{E}(U) &\triangleq \max_{i=0,1,\ldots,I-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} \big|U(t,X_t) - u(t,X_t)\big|^2\Big] \\ &= \max_{i=0,1,\ldots,I-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} \big|U(t,X_t) - Y_t\big|^2\Big].\end{aligned}$$

Similarly, we define

$$\begin{aligned}\mathcal{D}(U) &\triangleq \mathbb{E}\Big[\sum_{i=0}^{I-1}\int_{t_i}^{t_{i+1}} \big|\sigma(t,X_t)^\top \big(\mathrm{D}_x U(t,X_t) - \mathrm{D}_x u(t,X_t)\big)\big|^2 \mathrm{d}t\Big] \\ &= \mathbb{E}\Big[\sum_{i=0}^{I-1}\int_{t_i}^{t_{i+1}} \big|\sigma(t,X_t)^\top \mathrm{D}_x U(t,X_t) - Z_t\big|^2 \mathrm{d}t\Big].\end{aligned}$$

A natural measure of the **approximation error** is

$$\mathcal{E}(U) \triangleq \max_{i=0,1,\ldots,I-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} \big|U(t,X_t) - u(t,X_t)\big|^2\Big]$$
$$= \max_{i=0,1,\ldots,I-1} \mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]} \big|U(t,X_t) - Y_t\big|^2\Big].$$

Similarly, we define

$$\mathcal{D}(U) \triangleq \mathbb{E}\Big[\sum_{i=0}^{I-1}\int_{t_i}^{t_{i+1}} \big|\sigma(t,X_t)^\top \big(\mathrm{D}_x U(t,X_t) - \mathrm{D}_x u(t,X_t)\big)\big|^2 \mathrm{d}t\Big]$$
$$= \mathbb{E}\Big[\sum_{i=0}^{I-1}\int_{t_i}^{t_{i+1}} \big|\sigma(t,X_t)^\top \mathrm{D}_x U(t,X_t) - Z_t\big|^2 \mathrm{d}t\Big].$$

Note: We approximate u on all of $(0,T] \times \mathbb{R}^d$.

Convergence Theorem I

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathcal{E}(U) + \mathcal{D}(U) \leq Ch^{-1}\mathbb{L}(U) + Ch, \qquad U \in \mathcal{U}.$$

Convergence Theorem I

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathcal{E}(U) + \mathcal{D}(U) \le C h^{-1} \mathbb{L}(U) + Ch, \qquad U \in \mathcal{U}.$$

Remarks:

- Heuristically, the result implies that small loss means good approximation.

Convergence Theorem I

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathcal{E}(U) + \mathcal{D}(U) \leq Ch^{-1}\mathbb{L}(U) + Ch, \qquad U \in \mathcal{U}.$$

Remarks:

- Heuristically, the result implies that small loss means good approximation.
- The same result is true for Raissi's loss functional $\mathbb{L}^{\mathrm{R}}(U)$.

Convergence Theorem I

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathcal{E}(U) + \mathcal{D}(U) \leq C h^{-1} \mathbb{L}(U) + Ch, \qquad U \in \mathcal{U}.$$

Remarks:

- Heuristically, the result implies that small loss means good approximation.
- The same result is true for Raissi's loss functional $\mathbb{L}^{\mathrm{R}}(U)$.
- The estimate is "sharp".

Example: Consider the PDE

$$\begin{aligned} u_t(t,x) + u_x(t,x) + 1 &= 0, && (t,x) \in [0,T) \times \mathbb{R}, \\ u(T,x) &= 1, && x \in \mathbb{R}. \end{aligned}$$

The unique solution is given by

$$u(t,x) = T - t + 1, \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

Consider the approximating function $U \equiv 1$. Then

$$\mathbb{L}(U) = Ih^2 = Th \qquad \text{and} \qquad \mathcal{E}(U) = T^2 = Th^{-1}\mathbb{L}(U).$$

Example: Consider the PDE

$$\begin{aligned} u_t(t,x) + u_x(t,x) + 1 &= 0, \qquad && (t,x) \in [0,T) \times \mathbb{R}, \\ u(T,x) &= 1, && x \in \mathbb{R}. \end{aligned}$$

The unique solution is given by

$$u(t,x) = T - t + 1, \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

Consider the approximating function $U \equiv 1$. Then

$$\mathbb{L}(U) = Ih^2 = Th \qquad \text{and} \qquad \mathcal{E}(U) = T^2 = Th^{-1}\mathbb{L}(U).$$

Conclusion: The fact that the previous theorem is sharp is a consequence of

- us not imposing **additional structure** on the set $\mathcal{U}$ of approximating functions
- and the fact that our **estimate is uniform** in $\mathcal{U}$.

Convergence Theorem II

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathbb{L}(U) \leq C\|U - u\|^2_{\mathrm{C}^{1,2}} + Ch^2, \qquad U \in \mathcal{U}.$$

Convergence Theorem II

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathbb{L}(U) \leq C\|U - u\|^2_{\mathrm{C}^{1,2}} + Ch^2, \qquad U \in \mathcal{U}.$$

Remarks:

- If there are functions U which can approximate u well, their loss functional $\mathbb{L}(U)$ will be small and hence the error $\mathcal{E}(U) + \mathcal{D}(U)$ of this approximating function will be small as well.

Convergence Theorem II

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathbb{L}(U) \leq C\|U - u\|^2_{\mathrm{C}^{1,2}} + Ch^2, \qquad U \in \mathcal{U}.$$

Remarks:

- If there are functions U which can approximate u well, their loss functional $\mathbb{L}(U)$ will be small and hence the error $\mathcal{E}(U) + \mathcal{D}(U)$ of this approximating function will be small as well.
- In the paper we prove a version where the uniform distance $\|U - u\|^2_{\mathrm{C}^{1,2}}$ is only considered on a closed subset $K \subseteq \mathbb{R}^d$. This requires us to also use the probability of leaving K as another control.

Convergence Theorem II

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathbb{L}(U) \leq C\|U - u\|^2_{\mathrm{C}^{1,2}} + Ch^2, \qquad U \in \mathcal{U}.$$

Remarks:

- If there are functions U which can approximate u well, their loss functional $\mathbb{L}(U)$ will be small and hence the error $\mathcal{E}(U) + \mathcal{D}(U)$ of this approximating function will be small as well.
- In the paper we prove a version where the uniform distance $\|U - u\|^2_{\mathrm{C}^{1,2}}$ is only considered on a closed subset $K \subseteq \mathbb{R}^d$. This requires us to also use the probability of leaving K as another control.
- The same result is not true for Raissi's loss functional $\mathbb{L}^{\mathrm{R}}(U)$.

Convergence Theorem II

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathbb{L}(U) \leq C\|U - u\|^2_{\mathrm{C}^{1,2}} + Ch^2, \qquad U \in \mathcal{U}.$$

Remarks:

- If there are functions U which can approximate u well, their loss functional $\mathbb{L}(U)$ will be small and hence the error $\mathcal{E}(U) + \mathcal{D}(U)$ of this approximating function will be small as well.
- In the paper we prove a version where the uniform distance $\|U - u\|^2_{\mathrm{C}^{1,2}}$ is only considered on a closed subset $K \subseteq \mathbb{R}^d$. This requires us to also use the probability of leaving K as another control.
- The same result is not true for Raissi's loss functional $\mathbb{L}^{\mathrm{R}}(U)$.
- The estimate is "sharp".

Example: Consider the PDE

$$\begin{aligned} u_t(t,x) + \frac{1}{2} u_{xx}(t,x) &= 0, && (t,x) \in [0,T) \times \mathbb{R}, \\ u(T,x) &= \sin(x), && x \in \mathbb{R}. \end{aligned}$$

The unique solution is given by

$$u(t,x) = \sin(x) e^{-(T-t)/2}, \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

One can show that

$$\mathbb{L}(u) \geq \frac{T}{12} e^{-T} h^2 \qquad \text{and} \qquad \mathbb{L}^{\mathrm{R}}(u) \geq \frac{T}{4} e^{-T} h.$$

Example: Consider the PDE

$$u_t(t,x) + \frac{1}{2}u_{xx}(t,x) = 0, \qquad (t,x) \in [0,T) \times \mathbb{R},$$
$$u(T,x) = \sin(x), \qquad x \in \mathbb{R}.$$

The unique solution is given by

$$u(t,x) = \sin(x)e^{-(T-t)/2}, \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

One can show that

$$\mathbb{L}(u) \geq \frac{T}{12}e^{-T}h^2 \qquad \text{and} \qquad \mathbb{L}^{\mathrm{R}}(u) \geq \frac{T}{4}e^{-T}h.$$

Conclusion:

- Even if $u \in \mathcal{U}$, the result cannot be improved.
- Raissi's loss functional has a strictly smaller rate of convergence.

Convergence Theorem III

Let U_h^* a minimizer of

$$U \mapsto \mathbb{L}(U), \qquad U \in \mathcal{U}.$$

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) \leq Ch^{-1} \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + Ch.$$

Convergence Theorem III

Let U_h^* a minimizer of

$$U \mapsto \mathbb{L}(U), \qquad U \in \mathcal{U}.$$

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) \leq C h^{-1} \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + Ch.$$

Remarks:

- The best rate of convergence that we can achieve is of order $1/2$.

Convergence Theorem III

Let U_h^* a minimizer of

$$U \mapsto \mathbb{L}(U), \qquad U \in \mathcal{U}.$$

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) \leq Ch^{-1} \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + Ch.$$

Remarks:

- The best rate of convergence that we can achieve is of order $1/2$.
- Given h, the set $\mathcal{U}$ should be chosen such that

$$\inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} \leq Ch^2.$$

Convergence Theorem III

Let U_h^* a minimizer of

$$U \mapsto \mathbb{L}(U), \qquad U \in \mathcal{U}.$$

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) \leq Ch^{-1} \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + Ch.$$

Remarks:

- The best rate of convergence that we can achieve is of order $1/2$.
- Given h, the set $\mathcal{U}$ should be chosen such that

$$\inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} \leq Ch^2.$$

- We can enlarge the set $\mathcal{U}$ as long as the bound in $\mathcal{H}^{3,\gamma}$ remains unchanged.

Proof. For $\epsilon > 0$, let $U_\epsilon \in \mathcal{U}$ such that

$$\|U_\epsilon - u\|^2_{\mathrm{C}^{1,2}} \leq \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + \epsilon h.$$

Proof. For $\epsilon > 0$, let $U_\epsilon \in \mathcal{U}$ such that

$$\|U_\epsilon - u\|^2_{\mathrm{C}^{1,2}} \leq \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + \epsilon h.$$

Then

$$\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*)$$

Proof. For $\epsilon > 0$, let $U_\epsilon \in \mathcal{U}$ such that

$$\|U_\epsilon - u\|^2_{\mathrm{C}^{1,2}} \leq \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + \epsilon h.$$

Then

$$\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) \leq C h^{-1} \mathbb{L}(U_h^*) + Ch$$

Proof. For $\epsilon > 0$, let $U_\epsilon \in \mathcal{U}$ such that

$$\|U_\epsilon - u\|^2_{\mathrm{C}^{1,2}} \leq \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + \epsilon h.$$

Then

$$\begin{aligned} \mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) &\leq Ch^{-1}\mathbb{L}(U_h^*) + Ch \\ &\leq Ch^{-1}\mathbb{L}(U_\epsilon) + Ch \end{aligned}$$

Proof. For $\epsilon > 0$, let $U_\epsilon \in \mathcal{U}$ such that

$$\|U_\epsilon - u\|^2_{\mathrm{C}^{1,2}} \leq \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + \epsilon h.$$

Then

$$\begin{aligned}
\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) &\leq Ch^{-1}\mathbb{L}(U_h^*) + Ch \\
&\leq Ch^{-1}\mathbb{L}(U_\epsilon) + Ch \\
&\leq Ch^{-1}\left[C\|U_\epsilon - u\|^2_{\mathrm{C}^{1,2}} + Ch^2\right] + Ch
\end{aligned}$$

Proof. For $\epsilon > 0$, let $U_\epsilon \in \mathcal{U}$ such that

$$\|U_\epsilon - u\|^2_{\mathrm{C}^{1,2}} \le \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + \epsilon h.$$

Then

$$\begin{aligned}
\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) &\le Ch^{-1}\mathbb{L}(U_h^*) + Ch \\
&\le Ch^{-1}\mathbb{L}(U_\epsilon) + Ch \\
&\le Ch^{-1}\left[C\|U_\epsilon - u\|^2_{\mathrm{C}^{1,2}} + Ch^2\right] + Ch \\
&\le C^2 h^{-1} \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + (C^2 + C)h + \epsilon C^2.
\end{aligned}$$

Proof. For $\epsilon > 0$, let $U_\epsilon \in \mathcal{U}$ such that

$$\|U_\epsilon - u\|^2_{\mathrm{C}^{1,2}} \leq \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + \epsilon h.$$

Then

$$\begin{aligned}
\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) &\leq Ch^{-1}\mathbb{L}(U_h^*) + Ch \\
&\leq Ch^{-1}\mathbb{L}(U_\epsilon) + Ch \\
&\leq Ch^{-1}\left[C\|U_\epsilon - u\|^2_{\mathrm{C}^{1,2}} + Ch^2\right] + Ch \\
&\leq C^2h^{-1} \inf_{U \in \mathcal{U}} \|U - u\|^2_{\mathrm{C}^{1,2}} + (C^2 + C)h + \epsilon C^2.
\end{aligned}$$

Now send $\epsilon \downarrow 0$.

Some Final Remarks

Some final **observations**:

- While the constants do not depend on the dimension d of the state space, they do depend on the dimension n of the Brownian motion.

Some final **observations**:

- While the constants do not depend on the dimension d of the state space, they do depend on the dimension n of the Brownian motion.
- Using Raissi's loss functional, we only obtain the bound

$$\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) \leq Ch^{-1} \inf_{U \in \mathcal{U}} \|U - u\|_{\mathrm{C}^{1,2}}^2 + C,$$

  which does not converge to zero.

Some final **observations**:

- While the constants do not depend on the dimension d of the state space, they do depend on the dimension n of the Brownian motion.
- Using Raissi's loss functional, we only obtain the bound

$$\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) \leq Ch^{-1} \inf_{U \in \mathcal{U}} \|U - u\|_{\mathrm{C}^{1,2}}^2 + C,$$

  which does not converge to zero.
- This does not mean that Raissi's deep solver does not converge.

Some final **observations**:

- While the constants do not depend on the dimension d of the state space, they do depend on the dimension n of the Brownian motion.
- Using Raissi's loss functional, we only obtain the bound

$$\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) \leq Ch^{-1} \inf_{U \in \mathcal{U}} \|U - u\|_{\mathrm{C}^{1,2}}^2 + C,$$

  which does not converge to zero.
- This does not mean that Raissi's deep solver does not converge.
- There are many open problems: optimizing the loss functional, choice of approximating functions, Monte Carlo error, more general PDEs, ...

Some final **observations**:

- While the constants do not depend on the dimension d of the state space, they do depend on the dimension n of the Brownian motion.
- Using Raissi's loss functional, we only obtain the bound

$$\mathcal{E}(U_h^*) + \mathcal{D}(U_h^*) \leq Ch^{-1} \inf_{U \in \mathcal{U}} \|U - u\|_{\mathrm{C}^{1,2}}^2 + C,$$

  which does not converge to zero.
- This does not mean that Raissi's deep solver does not converge.
- There are many open problems: optimizing the loss functional, choice of approximating functions, Monte Carlo error, more general PDEs, ...
- The paper is on SSRN: https://ssrn.com/abstract=3981933