An Introduction to Reinforcement Learning

Part II: Classical Reinforcement Learning

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Preliminary Discussion

We want to solve the **Markov Decision Process**

$$\mathcal{V}(s) = \sup_{\pi} \mathbb{E}_s \Big[\sum_{n=0}^{\infty} \beta^n r\big(S_n^{\pi}, A_n^{\pi}\big) \Big].$$

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But maybe:

- We have a way to simulate the state process without knowing p
- Or we at least have a lot of data available

Before moving on, we need some notation. Recall the (optimal) **value function**

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Conclusions:

- Knowing Q, we can compute $\mathcal{V}$ and an optimal policy
- To compute $\mathcal{V}$ (resp. Q) via $\mathcal{V}_\pi$ (resp. Q_π), we need to perform two things
  - **Prediction**: Compute $\mathcal{V}_\pi$ (resp. Q_π) for a given policy π
  - **Control**: Find an optimal policy π^*

The Prediction Problem

First naive attempt at the prediction problem: **Monte Carlo**

$$\mathcal{V}_\pi(s) \approx v_\pi^M(s) \triangleq \frac{1}{M} \sum_{m=1}^{M} \sum_{n=0}^{\infty} \beta^n r(S_n^{\pi,m}, A_n^{\pi,m})$$

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with $\alpha_{M+1} \triangleq 1/(M+1)$. We write this more compactly as

$$v_\pi(s) \leftarrow v_\pi(s) + \alpha\Big[\sum_{n=0}^{\infty}\beta^n r(S_n, A_n) - v_\pi(s)\Big].$$

Summing up: For naive Monte Carlo, we estimate

$$\mathcal{V}_\pi(s) = \mathbb{E}_s\Big[\sum_{n=0}^{\infty} \beta^n r\big(S_n^\pi, A_n^\pi\big)\Big]$$

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This is referred to as (one-step) **Temporal Difference Prediction**.

In fact, if we use the temporal difference update rule, nothing stops us from using one sample path to update the estimate for all visited states, that is

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Exploration Requirement

The estimate q_π can only converge if $(S, A) = (S^\pi, A^\pi)$ visits each state-action pair (s, a) infinitely many times.

The Control Problem

The main idea behind the control problem is **policy improvement**. Let π, π' be two policies with

$$\mathcal{V}_\pi(s) \leq Q_\pi\big(s, \pi'(s)\big), \qquad s \in \mathbb{S}.$$

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Thus π' is better than π. Conversely, given π, we can improve it by choosing

$$\pi'(s) \in \arg\max_{a \in \mathbb{A}(s)} Q_\pi(s, a).$$

Given an initial policy π_0, we construct a sequence of policies $\{\pi_k\}_{k\in\mathbb{N}_0}$ by choosing

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which is improving in the sense that

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But then $\mathcal{V}_{\pi_k}$ solves the DPP, implying that $\mathcal{V}=\mathcal{V}_{\pi_k}$ and $\pi^*=\pi_k$ is **optimal** by the verification theorem.

Putting Everything Together

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The way out is to choose **ϵ-greedy actions**, that is, choose the greedy action with probability $1 - \epsilon$ and a random action with probability ϵ.

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Regarding convergence, we have to choose α and ϵ adaptively such that $\epsilon_n \downarrow 0$ and the learning rate satisfies the **Robbins-Monro condition**

$$\sum_{n=0}^{\infty} \alpha_n = \infty \qquad \text{and} \qquad \sum_{n=0}^{\infty} \alpha_n^2 < \infty.$$

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Q-Learning (Off-Policy Temporal Difference Control)

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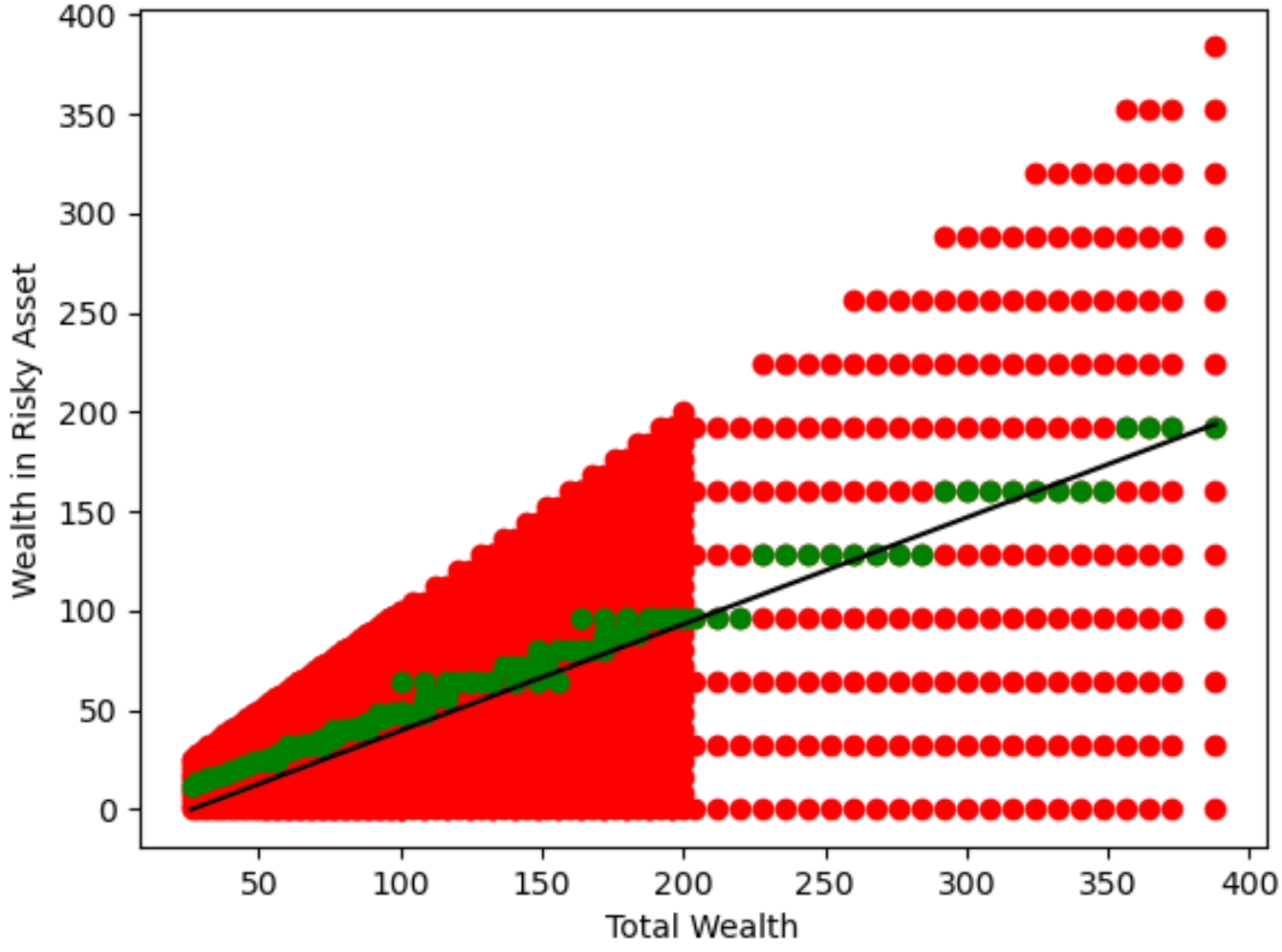
$$q(S_n, A_n) \leftarrow q(S_n, A_n) + \alpha \Big[r(S_n, A_n) + \beta \max_{a \in \mathbb{A}(s)} q(S_{n+1}, a) - q(S_n, A_n) \Big]$$

Numerical Experiments

Optimal investment with indivisible assets solved with **Dynamic Programming**



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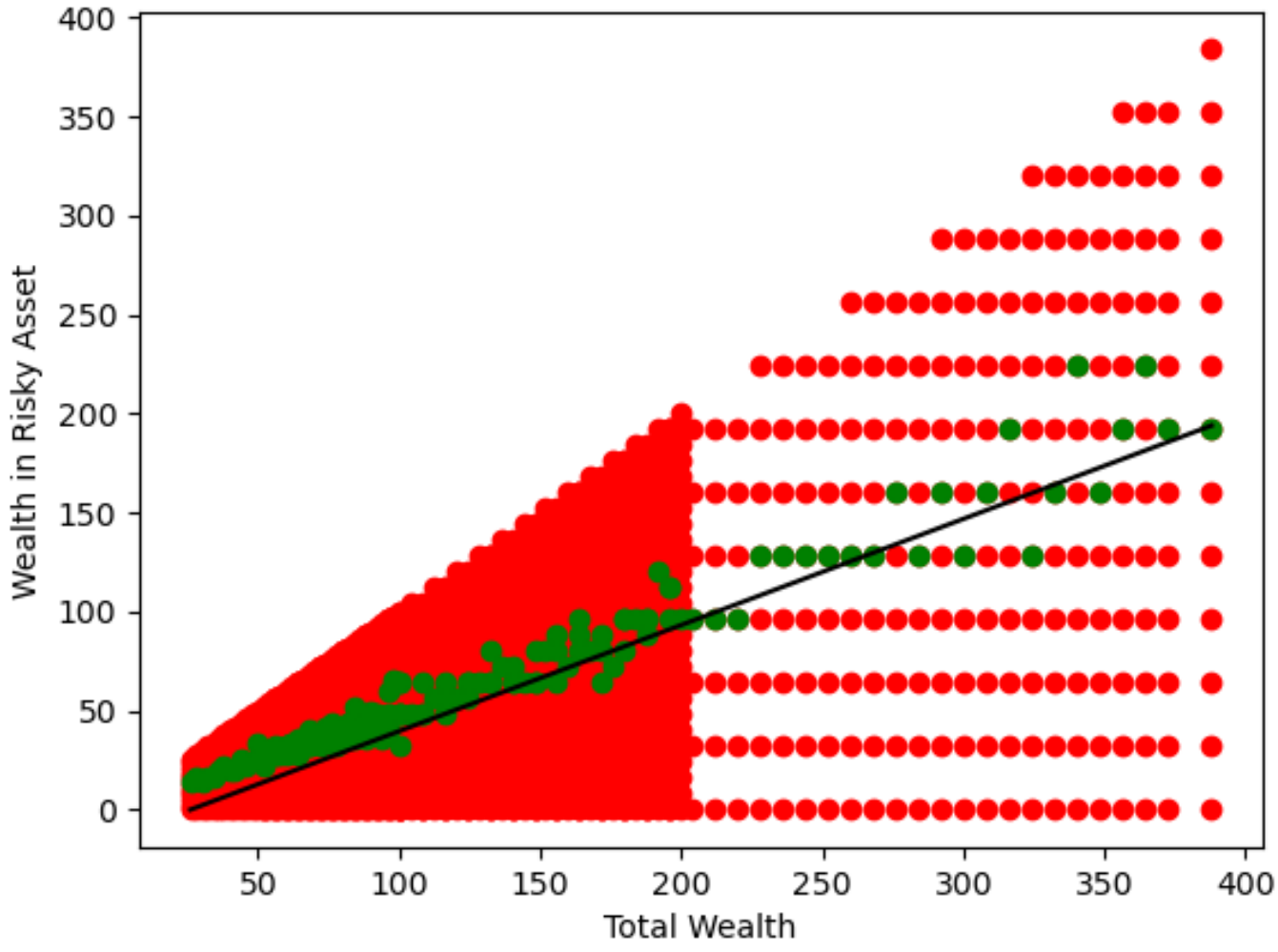

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- We need to be very careful with the choice of α
- Q-Learning does not require us to know the exact dynamics
- Where do we get all the data from?
- If we do not know the dynamics, how can we check convergence?