Principal-Agent Games of Optimal Stopping

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- Irreversible investment under uncertainty and with competition
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Transition from internal combustion engine vehicles to battery-electric vehicles.

More precisely:

- ▷ Irreversible investment under uncertainty and with competition
  $\longrightarrow$ **Game of Optimal Stopping**
- ▷ Socially optimal policy design and cost-benefit analysis
  $\longrightarrow$ **Principal-Agent Problem** with **Asymmetric Information**

Economic Background

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The Irreversible Investment Problem

Example for Consumer Preferences: Wright-Fisher Diffusion

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- Can be generalized significantly...

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- To wit, we consider $\mathcal{F}^X$-stopping times instead of $\mathcal{F}^W$-stopping times. Simultaneous stopping of both agents is possible!

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Then the stopping times associated with $D_1(0), D_1(1), D_2(0), D_2(1)$ form a Nash equilibrium.

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Time for some sketches!

The Principal-Agent Problem

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▷ Influences the reward structure of the agents:

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- Thus: α takes values in a compact subset $\mathbb{A}$ of $\mathbb{R}^d$ (for simplicity).

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$$\alpha \mapsto J_1^{\alpha}(x, z, \tau_1, \tau_2) = \mathbb{E}_{x,z}\Big[\int_0^{\tau_1} e^{-\rho_1 t}\tilde{h}_1\big(X_t, Z_t^i, \alpha\big)\mathrm{d}t + e^{-\rho_1 \tau_1}\tilde{f}_1(\alpha) + \int_{\tau_1}^{\infty} e^{-\rho_1 t}\tilde{g}_1\big(X_t, Z_t^2, \alpha\big)\mathrm{d}t\Big].$$

- Actions α include taxation and subsidies.
- Actions are static, i.e. implemented at time $t = 0$. In other words: X is assumed to be unobservable.
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- The policymaker benefits from a quick transition, but actions are costly.
- How to quantify response of the agents to a policy in the absence of uniqueness of Nash equilibria?

The Objective Functional:

- Each policy $\alpha \in \mathbb{A}$ gives rise to an optimal stopping game. Denote the associated set of Nash equilibria by $T(\alpha)$.

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&J_0(x, z_1, z_2, \alpha, \tau_1^*, \tau_2^*) \\
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- Otherwise, one has to replace the expectation in J_0 by a more general preference relation (work in progress!).

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- In the symmetric case, we are free to choose one representative equilibrium $(\tau_1^\alpha, \tau_2^\alpha) \in T(\alpha)$ to evaluate J_0.

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Optimal Policies

The mapping

$$\alpha \mapsto J_0\big(x, z_1, z_2, \alpha, \tau_1^\alpha, \tau_2^\alpha\big)$$

is continuous and hence there exists an optimal $\alpha^* = \alpha^*(x, z_1, z_2) \in \mathbb{A}$.

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- ▷ Numerical experiments with realistic data seem to indicate that first-movers are usually the largest agents or risk-seeking newcomers.
- ▷ Results appear to be quite sensitive to the choice of discount factors.
- ▷ Plethora of different effects, especially due to the non-uniqueness of equilibria. Much more to be said soon!

Conclusion:

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Thanks for your attention!