Convergence of Deep Solvers for Semilinear PDEs

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Joint work with Oliver Hager, Lotte Schnell, Charlotte Reimers and Maximilian Würschmidt

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- **independent** of the dimension of the state space
- but **dependent** on the dimension of the underlying Brownian motion.

Overview

(1) Derivation of the Deep Solver

(2) Main Convergence Results

(3) Some Final Remarks

Derivation of the Deep Solver

We consider a **partial differential equation** of the form

$$\begin{aligned} u_t + \mathcal{A}[u] + f(\cdot, u, \sigma^\top \mathrm{D}_x u) &= 0 && \text{on } [0,T) \times \mathbb{R}^d, \\ u(T,\cdot) &= g && \text{on } \mathbb{R}^d, \end{aligned}$$

where $\mathcal{A}$ is the second-order linear **differential operator** given by

$$\mathcal{A}[u] = \mu^\top \mathrm{D}_x u + \frac{1}{2} \operatorname{tr}\big[\sigma\sigma^\top \mathrm{D}_x^2 u\big].$$

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Recall that $\mathcal{A}$ is the **infinitesimal generator** of a diffusion with dynamics

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t.$$

Suppose that u is a **classical solution** of the PDE. By Itô's formula, we obtain

$$u(t, X_t) = u(T, X_T) - \int_t^T u_t(s, X_s) + \mathcal{A}[u](s, X_s)\mathrm{d}s \\ - \int_t^T \mathrm{D}_x u(s, X_s)^\top \sigma(s, X_s)\mathrm{d}W_s$$

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$$Y_s \triangleq u(s, X_s) \qquad \text{and} \qquad Z_s \triangleq \sigma(s, X_s)^\top \mathrm{D}_x u(s, X_s),$$

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and it follows that (Y, Z) is a solution of the **backward SDE**

$$Y_t = g(X_T) + \int_t^T f\big(s, X_s, Y_s, Z_s\big)\mathrm{d}s - \int_t^T Z_s^\top \mathrm{d}W_s.$$

For $I \in \mathbb{N}$, let $h = T/I$ denote the **step size** of the equidistant dissection

$$0 = t_0 < t_1 < \cdots < t_I = T$$

of $[0, T]$. Furthermore, write $\Delta W_{i+1} = W_{t_{i+1}} - W_{t_i}$ for the Brownian increments.

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Euler-Maruyama discretization of the forward SDE:

$$\begin{aligned} \bar{X}_0 &\triangleq x, \\ \bar{X}_{i+1} &\triangleq \bar{X}_i + \mu(t_i, \bar{X}_i)h + \sigma(t_i, \bar{X}_i)\Delta W_{i+1}, \qquad i = 0, 1, \ldots, I-1. \end{aligned}$$

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Naive Euler-Maruyama discretization of the backward SDE:

$$\begin{aligned} \bar{Y}_I &= g(\bar{X}_I), \\ \bar{Y}_{i+1} &= \bar{Y}_i - f\big(t_i, \bar{X}_i, \bar{Y}_i, \bar{Z}_i\big)h + \bar{Z}_i \Delta W_{i+1}, \qquad i = 0, 1, \ldots, I-1. \end{aligned}$$

Idea: A function U satisfies $U \approx u$ if and only if $U(t_i, \bar{X}_i) \approx \bar{Y}_i$ since

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$$\mathbb{L}^{\mathrm{R}}(U) \triangleq \mathbb{E}\left[\left|U(T, \bar{X}_I) - g(\bar{X}_I)\right|^2\right] + \sum_{i=0}^{I-1} \mathbb{E}\left[\left|\mathcal{L}_i^{\mathrm{R}}(U)\right|^2\right],$$

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Note: there are alternative loss functionals / approximation schemes based on similar ideas in the literature; see Beck et al. (2020) for an overview.

It turns out that Raissi's loss functional does not converge fast enough for our purposes, so we also consider the **adjusted loss functional**

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The correction term can be thought of as a consequence of a **Milstein discretization** instead of an Euler-Maruyama discretization of the backward SDE.

Main Convergence Results

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In light of the regularity of u, we assume that the **approximating functions** are chosen from a set

$$\mathcal{U} \subset \mathcal{H}^{3,\gamma} \text{ bounded.}$$

A natural measure for the **approximation error** is

$$\mathcal{E}(U) \triangleq \max_{i=0,1,\dots,I-1} \mathbb{E}\left[\sup_{t\in[t_i,t_{i+1}]} \left|U(t,X_t) - u(t,X_t)\right|^2\right]$$

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Convergence Theorem I

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

$$\mathcal{E}(U) + \mathcal{D}(U) \leq Ch^{-1}\mathbb{L}(U) + Ch, \qquad U \in \mathcal{U}.$$

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Remarks:

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- The same result is true for Raissi's loss functional $\mathbb{L}^{\mathrm{R}}(U)$.
- The estimate is "sharp".

Example: Consider the PDE

$$\begin{aligned} u_t(t,x) + u_x(t,x) + 1 &= 0, && (t,x) \in [0,T) \times \mathbb{R}, \\ u(T,x) &= 1, && x \in \mathbb{R}. \end{aligned}$$

The unique solution is given by

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Conclusion: The fact that the previous theorem is sharp is a consequence of

- us not imposing **additional structure** on the set $\mathcal{U}$ of approximating functions
- and the fact that our **estimate is uniform** in $\mathcal{U}$.

Convergence Theorem II

There exists a constant $C > 0$ which does not depend on the dimension d of the forward process X such that, for all h small enough,

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- The same result is not true for Raissi's loss functional $\mathbb{L}^{\mathrm{R}}(U)$.
- The estimate is "sharp".

Example: Consider the PDE

$$u_t(t,x) + \frac{1}{2}u_{xx}(t,x) = 0, \qquad (t,x) \in [0,T) \times \mathbb{R},$$

$$u(T,x) = \sin(x), \qquad x \in \mathbb{R}.$$

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- Even if $u \in \mathcal{U}$, the result cannot be improved.
- Raissi's loss functional has a lower order of convergence.

Convergence Theorem III

Let U_h^* a minimizer of

$$U \mapsto \mathbb{L}(U), \qquad U \in \mathcal{U}.$$

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- We can enlarge the set $\mathcal{U}$ as long as the bound in $\mathcal{H}^{3,\gamma}$ remains unchanged.

Proof. For $\epsilon > 0$, let $U_\epsilon \in \mathcal{U}$ such that

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Now send $\epsilon \downarrow 0$.

Some Final Remarks

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- In the paper, we work with general L^p functionals for $p \geq 2$.