Universität Trier

Option Pricing under Jump Uncertainty

Christoph Belak

Department IV – Mathematics
University of Trier
Germany

Joint work with **Olaf Menkens** (Dublin City University)

Mathematical Finance Seminar
Berlin, December 06, 2018

Outline

(1) Option Pricing under Jump Uncertainty: Model and Problem

(2) Constrained BSDE and PDE Characterizations

(4) Explicit Solutions and Numerical Results

Model and Problem Formulation

Problem Statement

Problem Statement: Option Pricing in the Presence of Uncertainty about Jumps in the Underlying

Problem Statement

Problem Statement: Option Pricing in the Presence of Uncertainty about Jumps in the Underlying

More precisely:

- We start with a **complete standard market** model $P = (P^0, P^1, \ldots, P^n)$.

Problem Statement

Problem Statement: Option Pricing in the Presence of Uncertainty about Jumps in the Underlying

More precisely:

- We start with a **complete standard market** model $P = (P^0, P^1, \dots, P^n)$.
- At stopping times τ, the **asset prices may jump** by fractions $\beta_1, \dots, \beta_n$, i.e.

$$P^i(\tau) = (1 + \beta_i) P^i(\tau-), \qquad i = 1, \dots, n.$$

Problem Statement

Problem Statement: Option Pricing in the Presence of Uncertainty about Jumps in the Underlying

More precisely:

- We start with a **complete standard market** model $P = (P^0, P^1, \dots, P^n)$.
- At stopping times τ, the **asset prices may jump** by fractions $\beta_1, \dots, \beta_n$, i.e.

$$P^i(\tau) = (1 + \beta_i) P^i(\tau-), \qquad i = 1, \dots, n.$$

- The seller of the option is uncertain about the **number**, the **timing**, the **direction**, and the **size** of the jumps.

Problem Statement

Problem Statement: Option Pricing in the Presence of Uncertainty about Jumps in the Underlying

More precisely:

- We start with a **complete standard market** model $P = (P^0, P^1, \dots, P^n)$.
- At stopping times τ, the **asset prices may jump** by fractions $\beta_1, \dots, \beta_n$, i.e.

$$P^i(\tau) = (1 + \beta_i) P^i(\tau-), \qquad i = 1, \dots, n.$$

- The seller of the option is uncertain about the **number**, the **timing**, the **direction**, and the **size** of the jumps.
- Instead, only **upper and lower bounds** on the number and size are given.

Problem Statement

Problem Statement: Option Pricing in the Presence of Uncertainty about Jumps in the Underlying

More precisely:

- We start with a **complete standard market** model $P = (P^0, P^1, \dots, P^n)$.
- At stopping times τ, the **asset prices may jump** by fractions $\beta_1, \dots, \beta_n$, i.e.

$$P^i(\tau) = (1 + \beta_i) P^i(\tau-), \qquad i = 1, \dots, n.$$

- The seller of the option is uncertain about the **number**, the **timing**, the **direction**, and the **size** of the jumps.
- Instead, only **upper and lower bounds** on the number and size are given.
- Options are priced taking a **worst-case superhedging perspective**.

Problem Statement

Problem Statement: Option Pricing in the Presence of Uncertainty about Jumps in the Underlying

More precisely:

- We start with a **complete standard market** model $P = (P^0, P^1, \dots, P^n)$.
- At stopping times τ, the **asset prices may jump** by fractions $\beta_1, \dots, \beta_n$, i.e.

$$P^i(\tau) = (1+\beta_i)P^i(\tau-), \qquad i = 1, \dots, n.$$

- The seller of the option is uncertain about the **number**, the **timing**, the **direction**, and the **size** of the jumps.
- Instead, only **upper and lower bounds** on the number and size are given.
- Options are priced taking a **worst-case superhedging perspective**.

Literature: Hua/Wilmott (1997), Mönnig (2012), Menkens (2016).

Main Results and Contributions

Main Results and Contributions:

- Generalization of the results of Mönnig (2012) to **non-Markovian** markets, **path-dependent** options, **American** options.

Main Results and Contributions:

- Generalization of the results of Mönnig (2012) to **non-Markovian** markets, **path-dependent** options, **American** options.
- Characterization of the price as the minimal solution of a **BSDE** satisfying a simultaneous **non-convex constraint** on the wealth and trading strategy.

Main Results and Contributions:

- Generalization of the results of Mönnig (2012) to **non-Markovian** markets, **path-dependent** options, **American** options.
- Characterization of the price as the minimal solution of a **BSDE** satisfying a simultaneous **non-convex constraint** on the wealth and trading strategy.
- In the special case of a Markovian market, characterization of the price as the unique viscosity solution of a fully nonlinear **pricing PDE** with a face-lifted terminal condition.

Main Results and Contributions

Main Results and Contributions:

- Generalization of the results of Mönnig (2012) to **non-Markovian** markets, **path-dependent** options, **American** options.
- Characterization of the price as the minimal solution of a **BSDE** satisfying a simultaneous **non-convex constraint** on the wealth and trading strategy.
- In the special case of a Markovian market, characterization of the price as the unique viscosity solution of a fully nonlinear **pricing PDE** with a face-lifted terminal condition.
- Numerical results for **American puts**.

Main Results and Contributions

Main Results and Contributions:

- Generalization of the results of Mönnig (2012) to **non-Markovian** markets, **path-dependent** options, **American** options.
- Characterization of the price as the minimal solution of a **BSDE** satisfying a simultaneous **non-convex constraint** on the wealth and trading strategy.
- In the special case of a Markovian market, characterization of the price as the unique viscosity solution of a fully nonlinear **pricing PDE** with a face-lifted terminal condition.
- Numerical results for **American puts**.
- In the sister paper Menkens (2016): **Explicit formulas** for prices and Greeks of European vanillas.

Wealth Dynamics / Options in the Absence of Jumps

We start with a **complete standard market**, in which the wealth X follows

$$\mathrm{d}X_t = \zeta_t^\top \sigma_t \theta_t \mathrm{d}t + \zeta_t^\top \sigma_t \mathrm{d}W_t, \qquad t \in [0, T], \qquad X_0 = x.$$

Wealth Dynamics / Options in the Absence of Jumps

We start with a **complete standard market**, in which the wealth X follows

$$\mathrm{d}X_t = \zeta_t^\top \sigma_t \theta_t \mathrm{d}t + \zeta_t^\top \sigma_t \mathrm{d}W_t, \qquad t \in [0,T], \qquad X_0 = x.$$

Here,

- ζ is the **trading strategy** (wealth invested in the respective assets),

Wealth Dynamics / Options in the Absence of Jumps

We start with a **complete standard market**, in which the wealth X follows

$$\mathrm{d}X_t = \zeta_t^\top \sigma_t \theta_t \mathrm{d}t + \zeta_t^\top \sigma_t \mathrm{d}W_t, \qquad t \in [0,T], \qquad X_0 = x.$$

Here,

- ζ is the **trading strategy** (wealth invested in the respective assets),
- θ is the unique bounded **market price of risk**, σ bounded **volatility matrix** with bounded inverse,

Wealth Dynamics / Options in the Absence of Jumps

We start with a **complete standard market**, in which the wealth X follows

$$\mathrm{d}X_t = \zeta_t^\top \sigma_t \theta_t \mathrm{d}t + \zeta_t^\top \sigma_t \mathrm{d}W_t, \qquad t \in [0,T], \qquad X_0 = x.$$

Here,

- ζ is the **trading strategy** (wealth invested in the respective assets),
- θ is the unique bounded **market price of risk**, σ bounded **volatility matrix** with bounded inverse,
- interest rates are set to **zero** (for simplicity).

Wealth Dynamics / Options in the Absence of Jumps

We start with a **complete standard market**, in which the wealth X follows

$$\mathrm{d}X_t = \zeta_t^\top \sigma_t \theta_t \mathrm{d}t + \zeta_t^\top \sigma_t \mathrm{d}W_t, \qquad t \in [0,T], \qquad X_0 = x.$$

Here,

- ζ is the **trading strategy** (wealth invested in the respective assets),
- θ is the unique bounded **market price of risk**, σ bounded **volatility matrix** with bounded inverse,
- interest rates are set to **zero** (for simplicity).

Options are positive processes ξ with $\mathbb{E}[\sup_{t\in[0,T]} |\xi_t|^2] < \infty$ such that either

Wealth Dynamics / Options in the Absence of Jumps

We start with a **complete standard market**, in which the wealth X follows

$$\mathrm{d}X_t = \zeta_t^\top \sigma_t \theta_t \mathrm{d}t + \zeta_t^\top \sigma_t \mathrm{d}W_t, \qquad t \in [0,T], \qquad X_0 = x.$$

Here,

- ζ is the **trading strategy** (wealth invested in the respective assets),
- θ is the unique bounded **market price of risk**, σ bounded **volatility matrix** with bounded inverse,
- interest rates are set to **zero** (for simplicity).

Options are positive processes ξ with $\mathbb{E}[\sup_{t\in[0,T]} |\xi_t|^2] < \infty$ such that either

- **European Option**: $\xi_t = 0$ for all $t \in [0,T)$, or

Wealth Dynamics / Options in the Absence of Jumps

We start with a **complete standard market**, in which the wealth X follows

$$\mathrm{d}X_t = \zeta_t^\top \sigma_t \theta_t \mathrm{d}t + \zeta_t^\top \sigma_t \mathrm{d}W_t, \qquad t \in [0,T], \qquad X_0 = x.$$

Here,

- ζ is the **trading strategy** (wealth invested in the respective assets),
- θ is the unique bounded **market price of risk**, σ bounded **volatility matrix** with bounded inverse,
- interest rates are set to **zero** (for simplicity).

Options are positive processes ξ with $\mathbb{E}[\sup_{t\in[0,T]} |\xi_t|^2] < \infty$ such that either

- **European Option**: $\xi_t = 0$ for all $t \in [0,T)$, or
- **American Option**: ξ is continuous.

Fair Price in the Absence of Jumps

The **fair price** $\mathcal{V}_0(\tau;\xi)$ of ξ at time τ is given by

$$\mathcal{V}_0(\tau;\xi) = Y_\tau,$$

where (Y, Z, K) is the **minimal solution of the BSDE**

$$\mathrm{d}Y_t = Z_t^\top \theta_t \mathrm{d}t + Z_t^\top \mathrm{d}W_t \qquad t \in [0,T], \qquad Y_T = \xi_T,$$

Literature: El Karoui/Peng/Quenez (1997), El Karoui/Pardoux/Peng (1997).

Fair Price in the Absence of Jumps

The **fair price** $\mathcal{V}_0(\tau;\xi)$ of ξ at time τ is given by

$$\mathcal{V}_0(\tau;\xi) = Y_\tau,$$

where (Y, Z, K) is the **minimal solution of the BSDE**

$$\mathrm{d}Y_t = Z_t^\top \theta_t \mathrm{d}t + Z_t^\top \mathrm{d}W_t - \mathrm{d}K_t, \qquad t \in [0,T], \qquad Y_T = \xi_T,$$

such that

$$Y_t \geq \xi_t, \qquad t \in [0,T], \qquad \text{and} \qquad \int_0^T [Y_t - \xi_t]\mathrm{d}K_t = 0.$$

Literature: El Karoui/Peng/Quenez (1997), El Karoui/Pardoux/Peng (1997).

Fair Price in the Absence of Jumps

The **fair price** $\mathcal{V}_0(\tau;\xi)$ of ξ at time τ is given by

$$\mathcal{V}_0(\tau;\xi) = Y_\tau,$$

where (Y, Z, K) is the **minimal solution of the BSDE**

$$\mathrm{d}Y_t = Z_t^\top \theta_t \mathrm{d}t + Z_t^\top \mathrm{d}W_t - \mathrm{d}K_t, \qquad t \in [0,T], \qquad Y_T = \xi_T,$$

such that

$$Y_t \geq \xi_t, \qquad t \in [0,T], \qquad \text{and} \qquad \int_0^T [Y_t - \xi_t] \mathrm{d}K_t = 0.$$

Remarks:

- Y and K are **continuous**, K is **increasing**, (Y, Z, K) is **square-integrable**.

Literature: El Karoui/Peng/Quenez (1997), El Karoui/Pardoux/Peng (1997).

Fair Price in the Absence of Jumps

The **fair price** $\mathcal{V}_0(\tau;\xi)$ of ξ at time τ is given by

$$\mathcal{V}_0(\tau;\xi) = Y_\tau,$$

where (Y,Z,K) is the **minimal solution of the BSDE**

$$\mathrm{d}Y_t = Z_t^\top \theta_t \mathrm{d}t + Z_t^\top \mathrm{d}W_t - \mathrm{d}K_t, \qquad t \in [0,T], \qquad Y_T = \xi_T,$$

such that

$$Y_t \geq \xi_t, \qquad t \in [0,T], \qquad \text{and} \qquad \int_0^T [Y_t - \xi_t] \mathrm{d}K_t = 0.$$

Remarks:

- Y and K are **continuous**, K is **increasing**, (Y,Z,K) is **square-integrable**.
- $K \equiv 0$ in the case of a **European option**.

Literature: El Karoui/Peng/Quenez (1997), El Karoui/Pardoux/Peng (1997).

Fair Price in the Absence of Jumps

The **fair price** $\mathcal{V}_0(\tau;\xi)$ of ξ at time τ is given by

$$\mathcal{V}_0(\tau;\xi) = Y_\tau,$$

where (Y, Z, K) is the **minimal solution of the BSDE**

$$\mathrm{d}Y_t = Z_t^\top \theta_t \mathrm{d}t + Z_t^\top \mathrm{d}W_t - \mathrm{d}K_t, \qquad t \in [0, T], \qquad Y_T = \xi_T,$$

such that

$$Y_t \geq \xi_t, \qquad t \in [0, T], \qquad \text{and} \qquad \int_0^T [Y_t - \xi_t] \mathrm{d}K_t = 0.$$

Remarks:

- Y and K are **continuous**, K is **increasing**, (Y, Z, K) is **square-integrable**.
- $K \equiv 0$ in the case of a **European option**.
- The **hedging strategy** is $\zeta = \sigma^{-\top} Z$.

Literature: El Karoui/Peng/Quenez (1997), El Karoui/Pardoux/Peng (1997).

Wealth Dynamics in the Presence of Jumps

In the case of a **single jump**, the wealth X follows

$$\mathrm{d}X_t = (\zeta_t^1)^\top \sigma_t \theta_t \mathrm{d}t + (\zeta_t^1)^\top \sigma_t \mathrm{d}W_t, \qquad t \in [0, \tau),$$

Here,

- ζ^1 is the strategy used **before the jump**,

Wealth Dynamics in the Presence of Jumps

In the case of a **single jump**, the wealth X follows

$$\mathrm{d}X_t = (\zeta_t^1)^\top \sigma_t \theta_t \mathrm{d}t + (\zeta_t^1)^\top \sigma_t \mathrm{d}W_t, \qquad t \in [0, \tau),$$
$$X_\tau = X_{\tau-} + \beta^\top \zeta_\tau^1, \qquad \text{on } \{\tau \leq T\},$$

Here,

- ζ^1 is the strategy used **before the jump**,
- $\vartheta = (\tau, \beta)$ is the **jump scenario** with $\beta \in [\underline{\beta}_\tau, \overline{\beta}_\tau]$,

Wealth Dynamics in the Presence of Jumps

In the case of a **single jump**, the wealth X follows

$$\begin{aligned} \mathrm{d}X_t &= (\zeta_t^1)^\top \sigma_t \theta_t \mathrm{d}t + (\zeta_t^1)^\top \sigma_t \mathrm{d}W_t, && t \in [0, \tau), \\ X_\tau &= X_{\tau-} + \beta^\top \zeta_\tau^1, && \text{on } \{\tau \le T\}, \end{aligned}$$

Here,

- ζ^1 is the strategy used **before the jump**,
- $\vartheta = (\tau, \beta)$ is the **jump scenario** with $\beta \in [\underline{\beta}_\tau, \overline{\beta}_\tau]$,
- the **minimum and maximum jump size** processes $-1 < \underline{\beta} \le 0$ and $0 \le \overline{\beta}$ are assumed to be continuous and bounded,

Wealth Dynamics in the Presence of Jumps

In the case of a **single jump**, the wealth X follows

$$\begin{aligned} \mathrm{d}X_t &= (\zeta_t^1)^\top \sigma_t \theta_t \mathrm{d}t + (\zeta_t^1)^\top \sigma_t \mathrm{d}W_t, && t \in [0, \tau), \\ X_\tau &= X_{\tau-} + \beta^\top \zeta_\tau^1, && \text{on } \{\tau \le T\}, \\ \mathrm{d}X_t &= (\zeta_t^\vartheta)^\top \sigma_t \theta_t \mathrm{d}t + (\zeta_t^\vartheta)^\top \sigma_t \mathrm{d}W_t, && t \in [\tau, T] \end{aligned}$$

Here,

- ζ^1 is the strategy used **before the jump**,
- $\vartheta = (\tau, \beta)$ is the **jump scenario** with $\beta \in [\underline{\beta}_\tau, \overline{\beta}_\tau]$,
- the **minimum and maximum jump size** processes $-1 < \underline{\beta} \le 0$ and $0 \le \overline{\beta}$ are assumed to be continuous and bounded,
- ζ^ϑ is the **strategy used after the jump** scenario ϑ (jump-dependent).

Wealth Dynamics in the Presence of Jumps

In the case of a **single jump**, the wealth X follows

$$\begin{aligned} \mathrm{d}X_t &= (\zeta_t^1)^\top \sigma_t \theta_t \mathrm{d}t + (\zeta_t^1)^\top \sigma_t \mathrm{d}W_t, && t \in [0, \tau), \\ X_\tau &= X_{\tau-} + \beta^\top \zeta_\tau^1, && \text{on } \{\tau \leq T\}, \\ \mathrm{d}X_t &= (\zeta_t^\vartheta)^\top \sigma_t \theta_t \mathrm{d}t + (\zeta_t^\vartheta)^\top \sigma_t \mathrm{d}W_t, && t \in [\tau, T] \end{aligned}$$

Here,

- ζ^1 is the strategy used **before the jump**,
- $\vartheta = (\tau, \beta)$ is the **jump scenario** with $\beta \in [\underline{\beta}_\tau, \overline{\beta}_\tau]$,
- the **minimum and maximum jump size** processes $-1 < \underline{\beta} \leq 0$ and $0 \leq \overline{\beta}$ are assumed to be continuous and bounded,
- ζ^ϑ is the **strategy used after the jump** scenario ϑ (jump-dependent).

Extensions: At most n **jumps**, **changing market parameters** after a jump (possibly jump-dependent).

Assumptions on Options

In the following, we impose the following assumptions:

- **European Option**: $\xi_T = \xi_E(P^\vartheta)$, where

$$\xi_E : \mathrm{D}\big([0,T], \mathbb{R}_+^{n+1}\big) \times \Omega \to [0,\infty)$$

  is measurable.

Assumptions on Options

In the following, we impose the following assumptions:

- **European Option**: $\xi_T = \xi_E(P^\vartheta)$, where

$$\xi_E : \mathrm{D}\big([0,T], \mathbb{R}_+^{n+1}\big) \times \Omega \to [0,\infty)$$

is measurable.

- **American Option**: $\xi_t = \xi_A(t, P_t^\vartheta)$, where

$$\xi_A : [0,T] \times \mathbb{R}_+^{n+1} \times \Omega \to [0,\infty)$$

is predictable and jointly continuous in the first two variables.

Assumptions on Options

We denote by $V_0(t, \beta, \omega)$ the **jump-free price** of ξ at time t in state ω, **conditional on the jump** (t, β) in the underlying.

In the following, we impose the following assumptions:

- **European Option**: $\xi_T = \xi_E(P^\vartheta)$, where

  $$\xi_E : \mathrm{D}\big([0,T], \mathbb{R}_+^{n+1}\big) \times \Omega \to [0, \infty)$$

  is measurable.

- **American Option**: $\xi_t = \xi_A(t, P_t^\vartheta)$, where

  $$\xi_A : [0,T] \times \mathbb{R}_+^{n+1} \times \Omega \to [0, \infty)$$

  is predictable and jointly continuous in the first two variables.

Assumptions on Options

We denote by $V_0(t,\beta,\omega)$ the **jump-free price** of ξ at time t in state ω, **conditional on the jump** (t,β) in the underlying.

In the following, we impose the following assumptions:

- **European Option**: $\xi_T = \xi_E(P^\vartheta)$, where

$$\xi_E : \mathrm{D}\big([0,T],\mathbb{R}_+^{n+1}\big) \times \Omega \to [0,\infty)$$

  is measurable.

- **American Option**: $\xi_t = \xi_A(t, P_t^\vartheta)$, where

$$\xi_A : [0,T] \times \mathbb{R}_+^{n+1} \times \Omega \to [0,\infty)$$

  is predictable and jointly continuous in the first two variables.

- $\mathcal{V}_0$ can be chosen such that the mapping $(t,\beta) \mapsto \mathcal{V}_0(t,\beta,\omega)$ is **jointly continuous** for all $\omega \in \Omega$.

Assumptions on Options

We denote by $V_0(t, \beta, \omega)$ the **jump-free price** of ξ at time t in state ω, **conditional on the jump** (t, β) in the underlying.

In the following, we impose the following assumptions:

- **European Option**: $\xi_T = \xi_E(P^\vartheta)$, where

  $$\xi_E : \mathrm{D}\big([0,T], \mathbb{R}_+^{n+1}\big) \times \Omega \to [0, \infty)$$

  is measurable.
- **American Option**: $\xi_t = \xi_A(t, P_t^\vartheta)$, where

  $$\xi_A : [0,T] \times \mathbb{R}_+^{n+1} \times \Omega \to [0, \infty)$$

  is predictable and jointly continuous in the first two variables.
- $\mathcal{V}_0$ can be chosen such that the mapping $(t, \beta) \mapsto \mathcal{V}_0(t, \beta, \omega)$ is **jointly continuous** for all $\omega \in \Omega$.

Remark: Sufficient conditions for the last assumption are available. The proof is based on the Kolmogorov-Centsov continuity theorem.

The Jump Uncertainty Price

Definition: Jump Uncertainty Price

The **jump uncertainty price** $\mathcal{V}_1$ at time $t = 0$ is defined as

$$\mathcal{V}_1 \triangleq \inf\Big\{ x \geq 0 : \text{there exists } (\zeta^1, \{\zeta^\vartheta\}_\vartheta) \text{ such that } X_t^{\zeta^1,\zeta^\vartheta,\vartheta} \geq \xi_t \text{ for all } t \in [0,T] \text{ and every jump scenario } \vartheta = (\tau, \beta) \Big\}.$$

In other words: The jump uncertainty price is the **smallest initial capital** required to **superhedge** the option in **every conceivable jump scenario** (including the no-jump scenario).

When is the Jump Uncertainty Price Finite?

At the time of a jump in the underlying, the **wealth jumps** as well:

$$X_\tau = X_{\tau-} + \beta^\top \zeta^1_\tau.$$

Hence ζ^1 needs to be **bounded to avoid bankruptcy**.

When is the Jump Uncertainty Price Finite?

At the time of a jump in the underlying, the **wealth jumps** as well:

$$X_\tau = X_{\tau-} + \beta^\top \zeta^1_\tau.$$

Hence ζ^1 needs to be **bounded to avoid bankruptcy**.

Intuitively, this means that only options which can be **superhedged with a bounded trading strategy** can have a finite jump uncertainty price.

When is the Jump Uncertainty Price Finite?

At the time of a jump in the underlying, the **wealth jumps** as well:

$$X_\tau = X_{\tau-} + \beta^\top \zeta^1_\tau.$$

Hence ζ^1 needs to be **bounded to avoid bankruptcy**.

Intuitively, this means that only options which can be **superhedged with a bounded trading strategy** can have a finite jump uncertainty price.

Proposition

Suppose that

$$\xi_E(P) \le C\big(1 + \textstyle\sum_{i=0}^n P_T^i\big)$$

in the European case or

$$\xi_A(t, P_t) \le C\big(1 + \textstyle\sum_{i=0}^n P_t^i\big)$$

in the American case. Then there exists at least one superhedging strategy and the jump uncertainty price is finite.

Constrained BSDE and PDE Characterizations

Characterization of the Set of Superhedging Strategies

The Main Idea

In order to superhedge the claim, the trader has to ensure that the wealth after a jump dominates the price in the jump-free market.

Characterization of the Set of Superhedging Strategies

The Main Idea

In order to superhedge the claim, the trader has to ensure that the wealth after a jump dominates the price in the jump-free market.

Mathematically, this means

$$H\left(\tau, X_{\tau-}^{x,\zeta_1}, \zeta_\tau^1\right) \geq 0 \qquad \text{for all stopping times } \tau, \tag{JC}$$

where

$$H(t,x,z) \triangleq \inf_{\beta \in [\underline{\beta}_t, \overline{\beta}_t]} \Big[x + \beta^\top z - \mathcal{V}_0(t,\beta) \Big].$$

Characterization of the Set of Superhedging Strategies

The Main Idea

In order to superhedge the claim, the trader has to ensure that the wealth after a jump dominates the price in the jump-free market.

Mathematically, this means

$$H\big(\tau, X^{x,\zeta_1}_{\tau-}, \zeta^1_\tau\big) \geq 0 \qquad \text{for all stopping times } \tau, \tag{JC}$$

where

$$H(t,x,z) \triangleq \inf_{\beta \in [\underline{\beta}_t, \overline{\beta}_t]} \Big[x + \beta^\top z - \mathcal{V}_0(t,\beta)\Big].$$

Theorem

There exists a superhedging strategy for ξ if and only if there exist x, ζ^1 satisfying the jump constraint (JC).

BSDE Characterization of the Jump Uncertainty Price

Given the characterization of the set of superhedging strategies, we now "merely" have to find (x, ζ^1) satisfying (JC) requiring the least initial wealth.

BSDE Characterization of the Jump Uncertainty Price

Given the characterization of the set of superhedging strategies, we now "merely" have to find (x, ζ^1) satisfying (JC) requiring the least initial wealth.

Theorem (BSDE Characterization)

Suppose that there exists at least one superhedging strategy for ξ. Then $\mathcal{V}_1 = Y_0$, where (Y, Z, C) is the minimal solution of the BSDE

$$\mathrm{d}Y_t = Z_t \theta_t \mathrm{d}t + Z_t \mathrm{d}W_t - \mathrm{d}C_t, \qquad t \in [0, T], \qquad Y_T = \xi_T$$

satisfying the constraint

$$H\big(t, Y_t, \sigma_t^{-\top} Z_t\big) \geq 0 \qquad \text{for all } t \in [0, T] \text{ almost surely.}$$

A corresponding pre-jump superhedging strategy is given by $\zeta^1 \triangleq \sigma^{-\top} Z$.

Idea of the Proof

The **proof** proceeds in several steps:

(1) Show that the existence of a superhedging strategy implies the existence of at least one solution of the BSDE satisfying the jump constraint (JC).

Idea of the Proof

The **proof** proceeds in several steps:

(1) Show that the existence of a superhedging strategy implies the existence of at least one solution of the BSDE satisfying the jump constraint (JC).

(2) Apply the results of Peng (1999) to obtain the existence of a minimal solution $(Y, \tilde{Z}, C)$ satisfying

$$H\big(t, Y_t, \sigma_t^{-\top} \tilde{Z}_t\big) \geq 0 \qquad \mathrm{d}t \otimes \mathbb{P}\text{-almost everywhere.}$$

Idea of the Proof

The **proof** proceeds in several steps:

(1) Show that the existence of a superhedging strategy implies the existence of at least one solution of the BSDE satisfying the jump constraint (JC).

(2) Apply the results of Peng (1999) to obtain the existence of a minimal solution $(Y, \tilde{Z}, C)$ satisfying

$$H\big(t, Y_t, \sigma_t^{-\top} \tilde{Z}_t\big) \geq 0 \qquad \mathrm{d}t \otimes \mathbb{P}\text{-almost everywhere.}$$

(3) Show that $\tilde{Z}$ admits a modification Z which satisfies

$$H\big(t, Y_t, \sigma_t^{-\top} Z_t\big) \geq 0 \qquad \text{for all } t \in [0, T] \text{ almost surely.}$$

Idea of the Proof

The **proof** proceeds in several steps:

(1) Show that the existence of a superhedging strategy implies the existence of at least one solution of the BSDE satisfying the jump constraint (JC).

(2) Apply the results of Peng (1999) to obtain the existence of a minimal solution $(Y, \tilde{Z}, C)$ satisfying

$$H\big(t, Y_t, \sigma_t^{-\top} \tilde{Z}_t\big) \geq 0 \qquad \mathrm{d}t \otimes \mathbb{P}\text{-almost everywhere.}$$

(3) Show that $\tilde{Z}$ admits a modification Z which satisfies

$$H\big(t, Y_t, \sigma_t^{-\top} Z_t\big) \geq 0 \qquad \text{for all } t \in [0, T] \text{ almost surely.}$$

(4) Show that $\mathcal{V}_1 = Y_0$.

The Markovian Market Model

Suppose now that we are in a **Markovian setting**, i.e. $\sigma, \theta, \underline{\beta}, \overline{\beta}, \xi$ are given in terms of deterministic functions of the prices of the underlyings.

The Markovian Market Model

Suppose now that we are in a **Markovian setting**, i.e. $\sigma, \theta, \underline{\beta}, \overline{\beta}, \xi$ are given in terms of deterministic functions of the prices of the underlyings.

In this setting, the fair jump-free price $\mathcal{V}_0$ and the jump uncertainty price $\mathcal{V}_1$ can be parametrized in terms of the **initial time** t and **initial prices** p of the underlyings, i.e.

$$\mathcal{V}_0 = \mathcal{V}_0(t, p) \qquad \text{and} \qquad \mathcal{V}_1 = \mathcal{V}_1(t, p).$$

The Markovian Market Model

Suppose now that we are in a **Markovian setting**, i.e. $\sigma, \theta, \underline{\beta}, \overline{\beta}, \xi$ are given in terms of deterministic functions of the prices of the underlyings.

In this setting, the fair jump-free price $\mathcal{V}_0$ and the jump uncertainty price $\mathcal{V}_1$ can be parametrized in terms of the **initial time** t and **initial prices** p of the underlyings, i.e.

$$\mathcal{V}_0 = \mathcal{V}_0(t, p) \qquad \text{and} \qquad \mathcal{V}_1 = \mathcal{V}_1(t, p).$$

Similarly, the **jump constraint** is given in terms of the deterministic function

$$H(t, p, x, z) \triangleq \inf_{\beta \in [\underline{\beta}(p), \overline{\beta}(p)]} \left[x + \beta^\top z - \mathcal{V}_0(t, \operatorname{diag}(\mathbf{1} + \beta) p) \right].$$

The Markovian Market Model

Suppose now that we are in a **Markovian setting**, i.e. $\sigma, \theta, \underline{\beta}, \overline{\beta}, \xi$ are given in terms of deterministic functions of the prices of the underlyings.

In this setting, the fair jump-free price $\mathcal{V}_0$ and the jump uncertainty price $\mathcal{V}_1$ can be parametrized in terms of the **initial time** t and **initial prices** p of the underlyings, i.e.

$$\mathcal{V}_0 = \mathcal{V}_0(t,p) \qquad \text{and} \qquad \mathcal{V}_1 = \mathcal{V}_1(t,p).$$

Similarly, the **jump constraint** is given in terms of the deterministic function

$$H(t,p,x,z) \triangleq \inf_{\beta \in [\underline{\beta}(p), \overline{\beta}(p)]} \left[x + \beta^\top z - \mathcal{V}_0(t, \operatorname{diag}(\mathbf{1} + \beta)p) \right].$$

Literature: Kharroubi/Ma/Pham/Zhang (2010), Peng/Xu (2013), Bouchard (2002), Bouchard/Elie/Touzi (2009).

PDE Characterization

We denote by $\mathcal{L}$ the infinitesimal generator of the P under the risk-neutral measure.

Theorem (PDE Characterization)

Suppose that ξ satisfies a linear growth condition. Then $\mathcal{V}_1$ is the unique continuous viscosity solution of

$$\min\Big\{-\partial_t \mathcal{V}_1(t,p) - \mathcal{L}\mathcal{V}_1(t,p), \\ H\Big(t, p, \mathcal{V}_1(t,p), \mathrm{diag}(p)\mathrm{D}_p\mathcal{V}_1(t,p)\Big)\Big\} = 0$$

PDE Characterization

We denote by $\mathcal{L}$ the infinitesimal generator of the P under the risk-neutral measure.

Theorem (PDE Characterization)

Suppose that ξ satisfies a linear growth condition. Then $\mathcal{V}_1$ is the unique continuous viscosity solution of

$$\min\Big\{-\partial_t\mathcal{V}_1(t,p) - \mathcal{L}\mathcal{V}_1(t,p),\\ H\big(t,p,\mathcal{V}_1(t,p),\mathrm{diag}(p)\mathrm{D}_p\mathcal{V}_1(t,p)\big)\Big\} = 0$$

such that $\mathcal{V}(T-,\cdot)$ satisfies the terminal condition

$$\min\Big\{\mathcal{V}_1(T-,p) - \xi(T,p),\\ H\big(T,p,\mathcal{V}_1(T-,p),\mathrm{diag}(p)\mathrm{D}_p\mathcal{V}_1(T-,p)\big)\Big\} = 0.$$

PDE Characterization

We denote by $\mathcal{L}$ the infinitesimal generator of the P under the risk-neutral measure.

Theorem (PDE Characterization)

Suppose that ξ satisfies a linear growth condition. Then $\mathcal{V}_1$ is the unique continuous viscosity solution of

$$\min\Big\{-\partial_t \mathcal{V}_1(t,p) - \mathcal{L}\mathcal{V}_1(t,p),\\ H\big(t,p,\mathcal{V}_1(t,p),\operatorname{diag}(p)\mathrm{D}_p\mathcal{V}_1(t,p)\big)\Big\} = 0$$

such that $\mathcal{V}(T-,\cdot)$ satisfies the terminal condition

$$\min\Big\{\mathcal{V}_1(T-,p) - \xi(T,p),\\ H\big(T,p,\mathcal{V}_1(T-,p),\operatorname{diag}(p)\mathrm{D}_p\mathcal{V}_1(T-,p)\big)\Big\} = 0.$$

Proof: Only interesting if you enjoy viscosity solutions as much as I do...

Explicit Solutions and Numerical Results

Explicit Solutions for European Vanilla Options

Menkens (2016) obtains explicit solutions for European calls and puts in a **Black-Scholes market**.

Explicit Solutions for European Vanilla Options

Menkens (2016) obtains explicit solutions for European calls and puts in a **Black-Scholes market**. For the put, the face-lifted terminal condition is given by

$$\mathcal{V}_1^{\mathrm{EP}}(T-,p) = [K-p]\mathbb{1}_{\{p\leq L\}}$$

Remark: The adjusted strike L is given by $L = K/[(1+\underline{\beta})(1+\overline{\beta})]$.

Explicit Solutions for European Vanilla Options

Menkens (2016) obtains explicit solutions for European calls and puts in a **Black-Scholes market**. For the put, the face-lifted terminal condition is given by

$$\mathcal{V}_1^{\mathrm{EP}}(T-, p) = [K - p]\mathbb{1}_{\{p \leq L\}} + \underline{\alpha} p^{-1/\underline{\beta}} \mathbb{1}_{\{p \leq L\}}$$

Remark: The adjusted strike L is given by $L = K/[(1+\underline{\beta})(1+\overline{\beta})]$.

Explicit Solutions for European Vanilla Options

Menkens (2016) obtains explicit solutions for European calls and puts in a **Black-Scholes market**. For the put, the face-lifted terminal condition is given by

$$\mathcal{V}_1^{\mathrm{EP}}(T-,p) = [K-p]\mathbb{1}_{\{p\le L\}} + \underline{\alpha}p^{-1/\underline{\beta}}\mathbb{1}_{\{p\le L\}} + \overline{\alpha}p^{-1/\overline{\beta}}\mathbb{1}_{\{p>L\}}.$$

Remark: The adjusted strike L is given by $L = K/[(1+\underline{\beta})(1+\overline{\beta})]$.

Explicit Solutions for European Vanilla Options

Menkens (2016) obtains explicit solutions for European calls and puts in a **Black-Scholes market**. For the put, the face-lifted terminal condition is given by

$$\mathcal{V}_1^{\mathrm{EP}}(T-,p) = \left[K - p\right]\mathbb{1}_{\{p\leq L\}} + \underline{\alpha} p^{-1/\underline{\beta}}\mathbb{1}_{\{p\leq L\}} + \overline{\alpha} p^{-1/\overline{\beta}}\mathbb{1}_{\{p>L\}}.$$

With this, the **jump uncertainty price** at time $t < T$ is given explicitly as

$$\mathcal{V}_1^{\mathrm{EP}}(t,p) = K\Phi\big(-d_2(L)\big) - p\Phi\big(-d_1(L)\big)$$

Remark: The adjusted strike L is given by $L = K/[(1+\underline{\beta})(1+\overline{\beta})]$.

Explicit Solutions for European Vanilla Options

Menkens (2016) obtains explicit solutions for European calls and puts in a **Black-Scholes market**. For the put, the face-lifted terminal condition is given by

$$\mathcal{V}_1^{\mathrm{EP}}(T-,p) = \big[K-p\big]\mathbb{1}_{\{p\le L\}} + \underline{\alpha} p^{-1/\underline{\beta}}\mathbb{1}_{\{p\le L\}} + \overline{\alpha} p^{-1/\overline{\beta}}\mathbb{1}_{\{p> L\}}.$$

With this, the **jump uncertainty price** at time $t<T$ is given explicitly as

$$\begin{aligned}\mathcal{V}_1^{\mathrm{EP}}(t,p) = K\Phi\big(-d_2(L)\big) - p\Phi\big(-d_1(L)\big) \\ + \underline{\alpha}\underline{\eta}(t)p^{1/\underline{\beta}}\Phi\big(-d_2(L)+\sigma\sqrt{T-t}/\underline{\beta}\big)\end{aligned}$$

Remark: The adjusted strike L is given by $L = K/[(1+\underline{\beta})(1+\overline{\beta})]$.

Explicit Solutions for European Vanilla Options

Menkens (2016) obtains explicit solutions for European calls and puts in a **Black-Scholes market**. For the put, the face-lifted terminal condition is given by

$$\mathcal{V}_1^{\mathrm{EP}}(T-,p) = \left[K-p\right]\mathbb{1}_{\{p\leq L\}} + \underline{\alpha}p^{-1/\underline{\beta}}\mathbb{1}_{\{p\leq L\}} + \overline{\alpha}p^{-1/\overline{\beta}}\mathbb{1}_{\{p>L\}}.$$

With this, the **jump uncertainty price** at time $t < T$ is given explicitly as

$$\begin{aligned}\mathcal{V}_1^{\mathrm{EP}}(t,p) = K\Phi\big(-d_2(L)\big) - p\Phi\big(-d_1(L)\big) \\ + \underline{\alpha}\underline{\eta}(t)p^{1/\underline{\beta}}\Phi\big(-d_2(L)+\sigma\sqrt{T-t}/\underline{\beta}\big) \\ + \overline{\alpha}\,\overline{\eta}(t)p^{1/\overline{\beta}}\Phi\big(d_2(L)-\sigma\sqrt{T-t}/\overline{\beta}\big).\end{aligned}$$

Remark: The adjusted strike L is given by $L = K/[(1+\underline{\beta})(1+\overline{\beta})]$.

Explicit Solutions for European Vanilla Options

Menkens (2016) obtains explicit solutions for European calls and puts in a **Black-Scholes market**. For the put, the face-lifted terminal condition is given by

$$\mathcal{V}_1^{\mathrm{EP}}(T-,p) = \big[K-p\big]\mathbb{1}_{\{p\le L\}} + \underline{\alpha} p^{-1/\underline{\beta}}\mathbb{1}_{\{p\le L\}} + \overline{\alpha} p^{-1/\overline{\beta}}\mathbb{1}_{\{p>L\}}.$$

With this, the **jump uncertainty price** at time $t<T$ is given explicitly as

$$\begin{aligned}\mathcal{V}_1^{\mathrm{EP}}(t,p) = K\Phi\big(-d_2(L)\big) - p\Phi\big(-d_1(L)\big) \\ + \underline{\alpha}\underline{\eta}(t)p^{1/\underline{\beta}}\Phi\big(-d_2(L)+\sigma\sqrt{T-t}/\underline{\beta}\big) \\ + \overline{\alpha}\,\overline{\eta}(t)p^{1/\overline{\beta}}\Phi\big(d_2(L)-\sigma\sqrt{T-t}/\overline{\beta}\big).\end{aligned}$$

Interestingly: The uncertainty premium $\mathcal{V}_1^{\mathrm{EP}} - \mathcal{V}_0^{\mathrm{EP}}$ at time $t<T$ is simply the (jump-free) Black-Scholes price of the uncertainty premium $\mathcal{V}_1^{\mathrm{EP}}(T-,p) - [K-p]^+$ at terminal time.

Remark: The adjusted strike L is given by $L = K/[(1+\underline{\beta})(1+\overline{\beta})]$.

Jump Uncertainty Prices of American Vanillas

Well known: In the absence of jump uncertainty, the price $\mathcal{V}_0^{\mathrm{AC}}$ of the American call coincides with the price of the European call $\mathcal{V}_0^{\mathrm{EC}}$ if interest rates are positive.

Jump Uncertainty Prices of American Vanillas

Well known: In the absence of jump uncertainty, the price $\mathcal{V}_0^{\mathrm{AC}}$ of the American call coincides with the price of the European call $\mathcal{V}_0^{\mathrm{EC}}$ if interest rates are positive.

This implies: The corresponding jump uncertainty prices $\mathcal{V}_1^{\mathrm{AC}}$ and $\mathcal{V}_1^{\mathrm{EC}}$ satisfy the same PDE, i.e.

$$\mathcal{V}_1^{\mathrm{AC}} = \mathcal{V}_1^{\mathrm{EC}}.$$

Jump Uncertainty Prices of American Vanillas

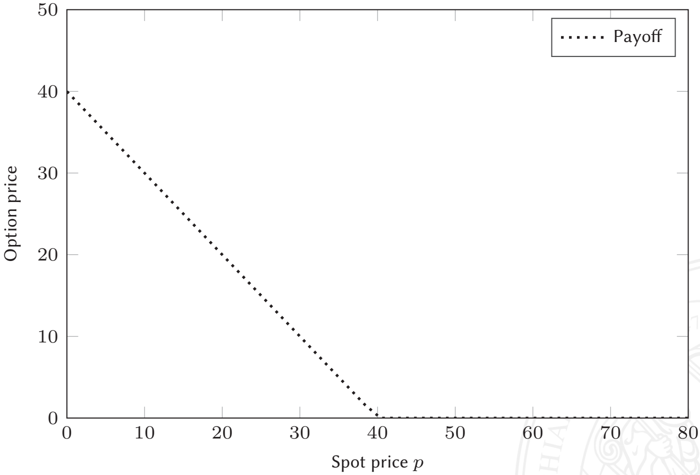
Well known: In the absence of jump uncertainty, the price $\mathcal{V}_0^{\mathrm{AC}}$ of the American call coincides with the price of the European call $\mathcal{V}_0^{\mathrm{EC}}$ if interest rates are positive.

This implies: The corresponding jump uncertainty prices $\mathcal{V}_1^{\mathrm{AC}}$ and $\mathcal{V}_1^{\mathrm{EC}}$ satisfy the same PDE, i.e.

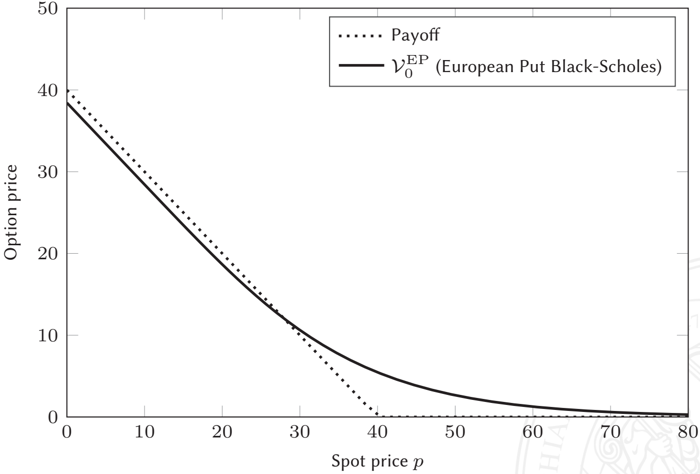
$$\mathcal{V}_1^{\mathrm{AC}} = \mathcal{V}_1^{\mathrm{EC}}.$$

The jump uncertainty price of the **American put**, on the other hand, differs from that of the European put and has to be computed **numerically**.

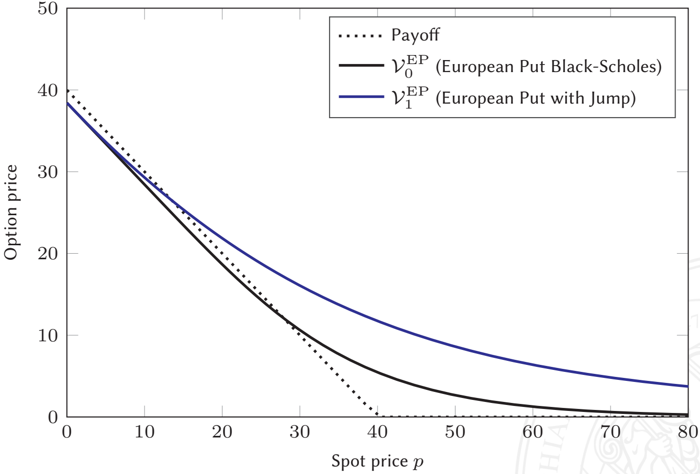
Numerical Results for the American Put



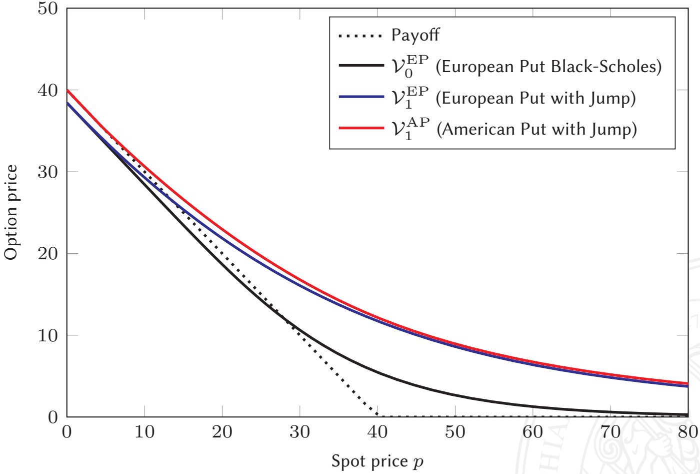
Numerical Results for the American Put



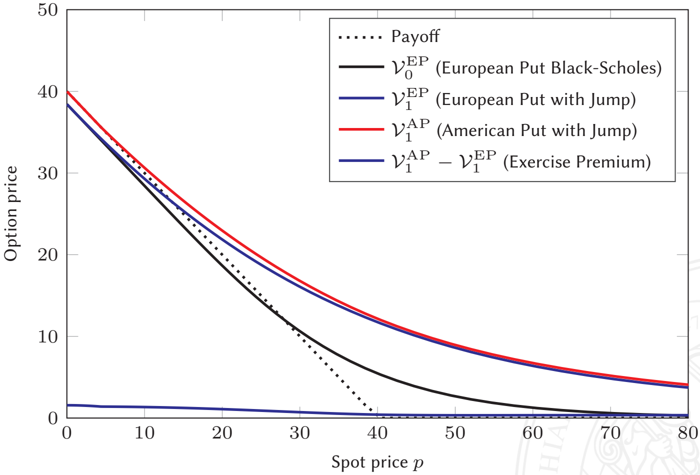
Numerical Results for the American Put



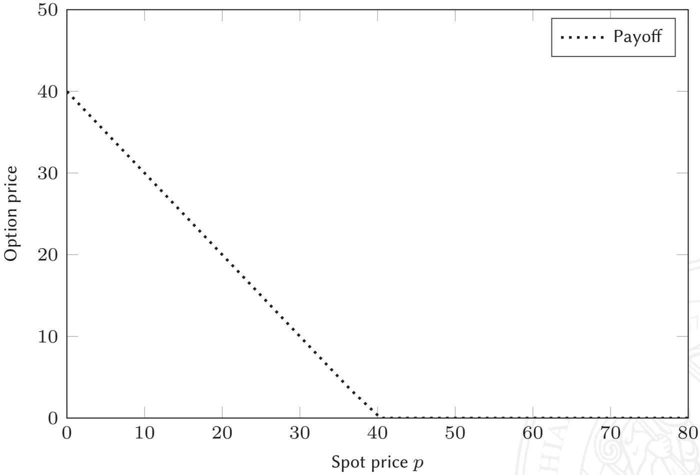
Numerical Results for the American Put



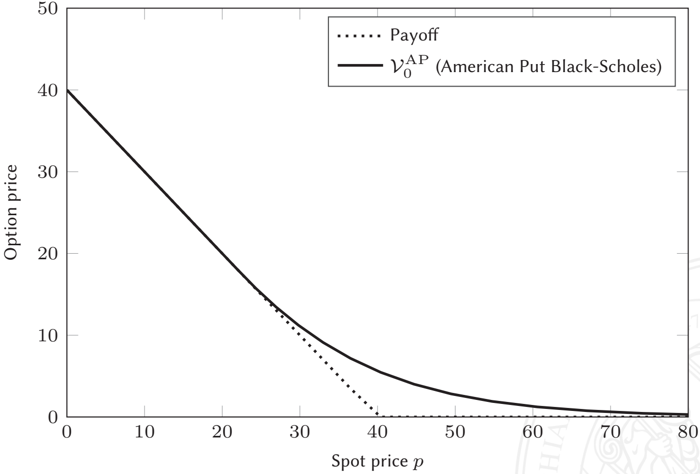
Numerical Results for the American Put



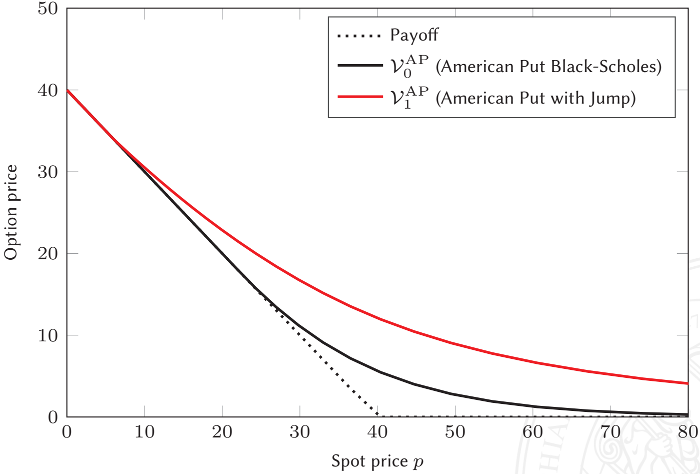
Numerical Results for the American Put



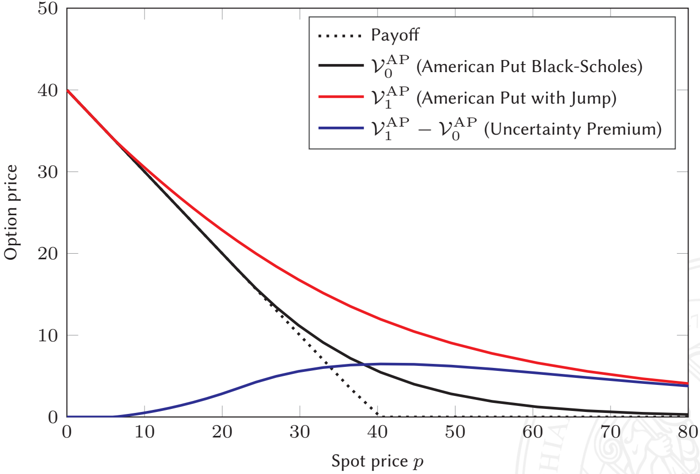
Numerical Results for the American Put



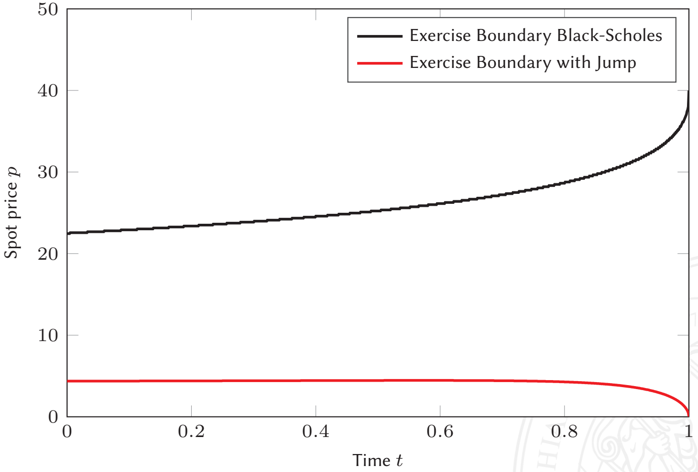
Numerical Results for the American Put



Numerical Results for the American Put



Exercise Boundaries for the American Put



Conclusion

- We consider the option pricing problem under **model uncertainty** with respect to jumps in the underlying assets.

Conclusion

- We consider the option pricing problem under **model uncertainty** with respect to jumps in the underlying assets.
- We extend the existing results to **non-Markovian**, **multidimensional** markets and **path-dependent** and **American** options.

Conclusion

- We consider the option pricing problem under **model uncertainty** with respect to jumps in the underlying assets.
- We extend the existing results to **non-Markovian**, **multidimensional** markets and **path-dependent** and **American** options.
- The **number**, **timing**, **direction**, and **size** of jumps is unknown. The bounds are possibly **path- and state-dependent**.

Conclusion

- We consider the option pricing problem under **model uncertainty** with respect to jumps in the underlying assets.
- We extend the existing results to **non-Markovian**, **multidimensional** markets and **path-dependent** and **American** options.
- The **number**, **timing**, **direction**, and **size** of jumps is unknown. The bounds are possibly **path- and state-dependent**.
- The results can be generalized to **jump-dependent changes in the market parameters**.

Conclusion

- We consider the option pricing problem under **model uncertainty** with respect to jumps in the underlying assets.
- We extend the existing results to **non-Markovian**, **multidimensional** markets and **path-dependent** and **American** options.
- The **number**, **timing**, **direction**, and **size** of jumps is unknown. The bounds are possibly **path- and state-dependent**.
- The results can be generalized to **jump-dependent changes in the market parameters**.
- The worst-case superhedging price is characterized as the minimal solution of a BSDE involving a simultaneous constraint on the wealth and the trading strategy.

Conclusion

- We consider the option pricing problem under **model uncertainty** with respect to jumps in the underlying assets.
- We extend the existing results to **non-Markovian**, **multidimensional** markets and **path-dependent** and **American** options.
- The **number**, **timing**, **direction**, and **size** of jumps is unknown. The bounds are possibly **path- and state-dependent**.
- The results can be generalized to **jump-dependent changes in the market parameters**.
- The worst-case superhedging price is characterized as the minimal solution of a BSDE involving a simultaneous constraint on the wealth and the trading strategy.
- In the Markovian special case, we derive a **nonlinear pricing PDE** with face-lifted terminal condition.

Conclusion

- We consider the option pricing problem under **model uncertainty** with respect to jumps in the underlying assets.
- We extend the existing results to **non-Markovian**, **multidimensional** markets and **path-dependent** and **American** options.
- The **number**, **timing**, **direction**, and **size** of jumps is unknown. The bounds are possibly **path- and state-dependent**.
- The results can be generalized to **jump-dependent changes in the market parameters**.
- The worst-case superhedging price is characterized as the minimal solution of a BSDE involving a simultaneous constraint on the wealth and the trading strategy.
- In the Markovian special case, we derive a **nonlinear pricing PDE** with face-lifted terminal condition.
- We study the **American put numerically**.

Conclusion

- We consider the option pricing problem under **model uncertainty** with respect to jumps in the underlying assets.
- We extend the existing results to **non-Markovian**, **multidimensional** markets and **path-dependent** and **American** options.
- The **number**, **timing**, **direction**, and **size** of jumps is unknown. The bounds are possibly **path- and state-dependent**.
- The results can be generalized to **jump-dependent changes in the market parameters**.
- The worst-case superhedging price is characterized as the minimal solution of a BSDE involving a simultaneous constraint on the wealth and the trading strategy.
- In the Markovian special case, we derive a **nonlinear pricing PDE** with face-lifted terminal condition.
- We study the **American put numerically**.

Thanks for listening!