Universität Trier

Utility Maximization with Constant Costs

Christoph Belak

Department IV – Mathematics
University of Trier
Germany

Based on joint work with **Sören Christensen** and **Frank T. Seifried**.

Larry Shepp Symposium on Optimal Stopping
Rice University, Houston, Texas
June 29, 2018

Outline

Utility Maximization with Constant Costs

The Market Model

We assume that the **portfolio** $X = \{X(t)\}_{t\in[0,T]}$ evolves as

$$\begin{aligned} \mathrm{d}X_1(t) &= rX_1(t)\mathrm{d}t, && t \in [\tau_k, \tau_{k+1}), \\ \mathrm{d}X_2(t) &= \mu X_2(t)\mathrm{d}t + \sigma X_2(t)\mathrm{d}W(t), && t \in [\tau_k, \tau_{k+1}), \end{aligned}$$

Remark: The model can be generalized (more assets, factor processes, short selling).

The Market Model

We assume that the **portfolio** $X = \{X(t)\}_{t\in[0,T]}$ evolves as

$$\begin{aligned}
\mathrm{d}X_1(t) &= rX_1(t)\mathrm{d}t, && t \in [\tau_k, \tau_{k+1}),\\
\mathrm{d}X_2(t) &= \mu X_2(t)\mathrm{d}t + \sigma X_2(t)\mathrm{d}W(t), && t \in [\tau_k, \tau_{k+1}),\\
X_1(\tau_k) &= X_1(\tau_k-) - \Delta_k\\
X_2(\tau_k) &= X_2(\tau_k-) + \Delta_k,
\end{aligned}$$

Remark: The model can be generalized (more assets, factor processes, short selling).

The Market Model

We assume that the **portfolio** $X = \{X(t)\}_{t\in[0,T]}$ evolves as

$$\begin{aligned}
\mathrm{d}X_1(t) &= rX_1(t)\mathrm{d}t, && t \in [\tau_k, \tau_{k+1}),\\
\mathrm{d}X_2(t) &= \mu X_2(t)\mathrm{d}t + \sigma X_2(t)\mathrm{d}W(t), && t \in [\tau_k, \tau_{k+1}),\\
X_1(\tau_k) &= X_1(\tau_k-) - \Delta_k - \gamma|\Delta_k| - C,\\
X_2(\tau_k) &= X_2(\tau_k-) + \Delta_k,
\end{aligned}$$

where $\gamma \in (0,1)$ (**proportional cost**) and $C > 0$ (**constant cost**).

Remark: The model can be generalized (more assets, factor processes, short selling).

The Market Model

We assume that the **portfolio** $X = \{X(t)\}_{t\in[0,T]}$ evolves as

$$\begin{aligned} dX_1(t) &= rX_1(t)dt, & t \in [\tau_k, \tau_{k+1}), \\ dX_2(t) &= \mu X_2(t)dt + \sigma X_2(t)dW(t), & t \in [\tau_k, \tau_{k+1}), \\ X_1(\tau_k) &= X_1(\tau_k-) - \Delta_k - \gamma|\Delta_k| - C, \\ X_2(\tau_k) &= X_2(\tau_k-) + \Delta_k, \end{aligned}$$

where $\gamma \in (0,1)$ (**proportional cost**) and $C > 0$ (**constant cost**).

We **prohibit short selling** of the stock.

Remark: The model can be generalized (more assets, factor processes, short selling).

The Market Model

We assume that the **portfolio** $X = \{X(t)\}_{t\in[0,T]}$ evolves as

$$\begin{aligned} \mathrm{d}X_1(t) &= rX_1(t)\mathrm{d}t, & t &\in [\tau_k, \tau_{k+1}), \\ \mathrm{d}X_2(t) &= \mu X_2(t)\mathrm{d}t + \sigma X_2(t)\mathrm{d}W(t), & t &\in [\tau_k, \tau_{k+1}), \\ X_1(\tau_k) &= X_1(\tau_k-) - \Delta_k - \gamma|\Delta_k| - C, \\ X_2(\tau_k) &= X_2(\tau_k-) + \Delta_k, \end{aligned}$$

where $\gamma \in (0,1)$ (**proportional cost**) and $C > 0$ (**constant cost**).

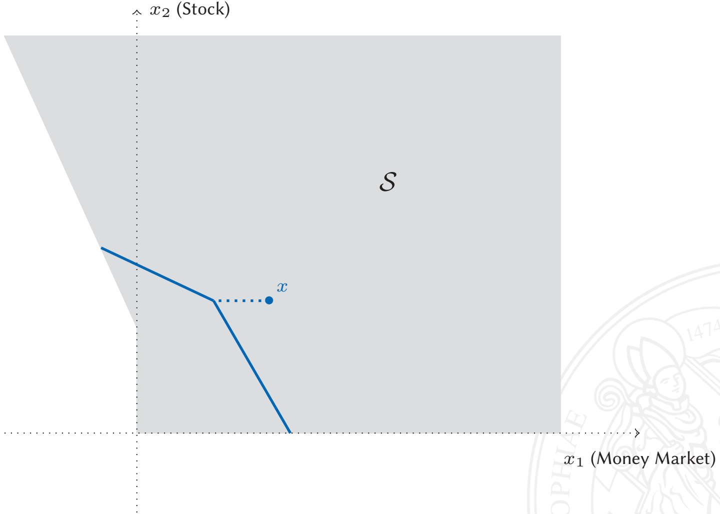
We **prohibit short selling** of the stock. A portfolio $x \in \mathbb{R} \times [0,\infty)$ is **solvent** if it has a positive liquidation value $\mathrm{L}(x)$, i.e.,

$$\mathrm{L}(x) \triangleq x_1 + \left(x_2 - \gamma x_2 - C\right)^+ > 0.$$

The set $\mathcal{S} \subset \mathbb{R} \times [0,\infty)$ of solvent portfolios is called the **solvency region**.

Remark: The model can be generalized (more assets, factor processes, short selling).

The Solvency Region



The Optimization Criterion

Now fix a **utility function** $U : \mathbb{R}_+ \to \mathbb{R}$ such that

- U is strictly increasing, continuous, and concave,
- U is lower bounded; without loss of generality $U(0) = 0$,
- U satisfies $U(\ell) \leq M(1 + |\ell|^p)$ for some $M > 0$, $p \in (0, 1)$.

The objective is to **maximize expected utility of terminal wealth**, i.e.

$$\mathcal{V}(t,x) = \sup_{\Lambda \in \mathcal{A}(t,x)} \mathbb{E}\Big[U\Big(\mathrm{L}\big(X^{\Lambda}_{t,x}(T)\big)\Big)\Big],$$

where $\mathcal{A}(t,x)$ denotes the set of **admissible strategies** $\Lambda = \{(\tau_k, \Delta_k)\}_{k \in \mathbb{N}}$ for the initial state (t,x), i.e. the set of strategies Λ for which

$$\mathrm{L}\big(X^{\Lambda}_{t,x}\big) \geq 0 \qquad \text{on } [t,T].$$

The Quasi-Variational Inequalities

The value function $\mathcal{V}$ is expected to be linked to the following **quasi-variational inequalities** (QVIs):

$$\min\left\{-\partial_t \mathcal{V}(t,x) - \mathcal{L}\mathcal{V}(t,x), \mathcal{V}(t,x) - \mathcal{M}\mathcal{V}(t,x)\right\} = 0, \quad (t,x) \in [0,T) \times \mathcal{S},$$

where $\mathcal{L}$ denotes the **infinitesimal generator** of the uncontrolled portfolio process given by

$$\mathcal{L}\mathcal{V}(t,x) \triangleq r x_1 \partial_{x_1} \mathcal{V}(t,x) + \mu x_2 \partial_{x_2} \mathcal{V}(t,x) + \frac{1}{2}\sigma^2 x_2^2 \partial^2_{x_2,x_2} \mathcal{V}(t,x),$$

The Quasi-Variational Inequalities

The value function $\mathcal{V}$ is expected to be linked to the following **quasi-variational inequalities** (QVIs):

$$\min\Big\{-\partial_t \mathcal{V}(t,x) - \mathcal{L}\mathcal{V}(t,x), \mathcal{V}(t,x) - \mathcal{M}\mathcal{V}(t,x)\Big\} = 0, \quad (t,x) \in [0,T) \times \mathcal{S},$$

where $\mathcal{L}$ denotes the **infinitesimal generator** of the uncontrolled portfolio process given by

$$\mathcal{L}\mathcal{V}(t,x) \triangleq r x_1 \partial_{x_1} \mathcal{V}(t,x) + \mu x_2 \partial_{x_2} \mathcal{V}(t,x) + \frac{1}{2}\sigma^2 x_2^2 \partial^2_{x_2,x_2} \mathcal{V}(t,x),$$

and $\mathcal{M}$ is the **maximum operator** given by

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta} \mathcal{V}(t, x_1 - \Delta - \gamma|\Delta| - C, x_2 + \Delta).$$

Related Literature

In the existing literature, there are four different approaches:

- **Classical PDE Solutions**: Eastham/Hastings (1988), Korn (1998), Bielecki/Pliska (2000), Liu (2004).
- **Viscosity Solutions**: Oksendal/Sulem (2002), Altarovici/Reppen/Soner (2016)
- **Iterated Optimal Stopping**: Korn (1998), Palczewski/Stettner (2007)
- **Small Cost Asymptotics**: Korn (1998), Korn/Laue (2002), Altarovici/Muhle-Karbe/Soner (2015), Feodoria/Kallsen (2016), Altarovici/Reppen/Soner (2016).

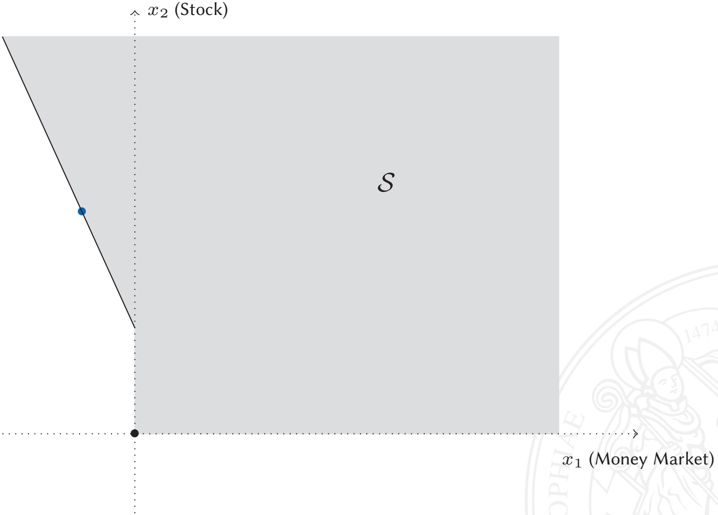
Related Literature

In the existing literature, there are four different approaches:

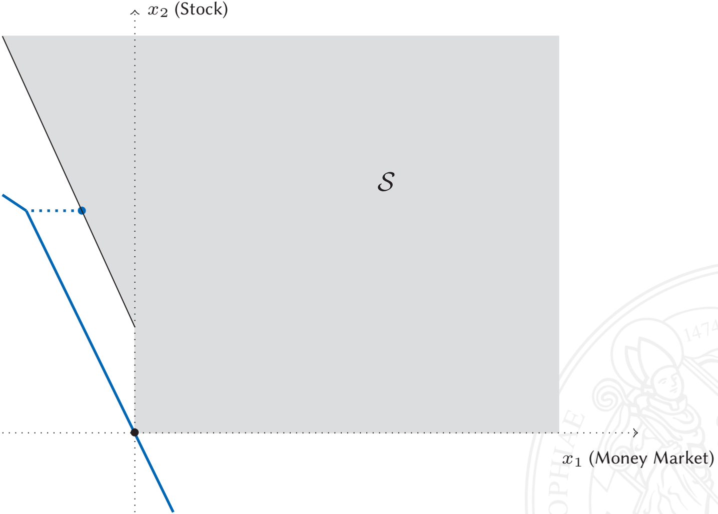
- **Classical PDE Solutions**: Eastham/Hastings (1988), Korn (1998), Bielecki/Pliska (2000), Liu (2004).
- **Viscosity Solutions**: Oksendal/Sulem (2002), Altarovici/Reppen/Soner (2016)
- **Iterated Optimal Stopping**: Korn (1998), Palczewski/Stettner (2007)
- **Small Cost Asymptotics**: Korn (1998), Korn/Laue (2002), Altarovici/Muhle-Karbe/Soner (2015), Feodoria/Kallsen (2016), Altarovici/Reppen/Soner (2016).

(Dis-)Continuity of the Value Function

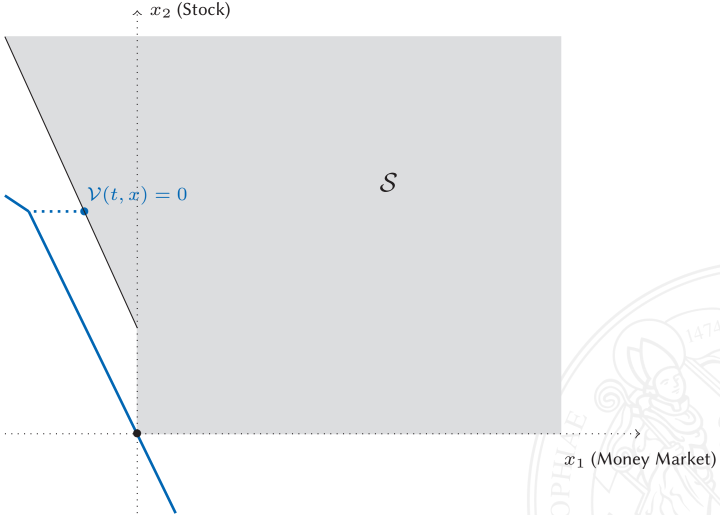
Discontinuity of the Value Function



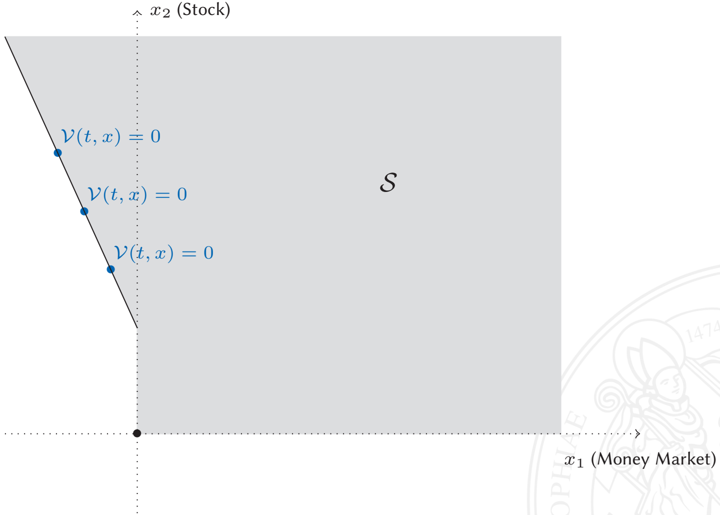
Discontinuity of the Value Function



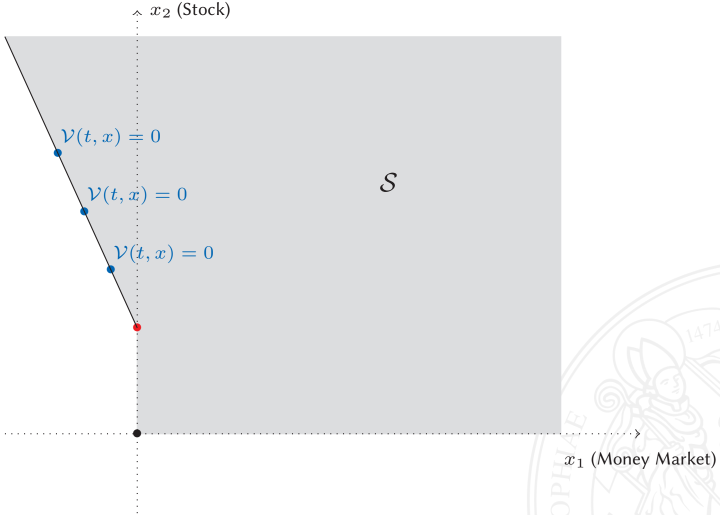
Discontinuity of the Value Function



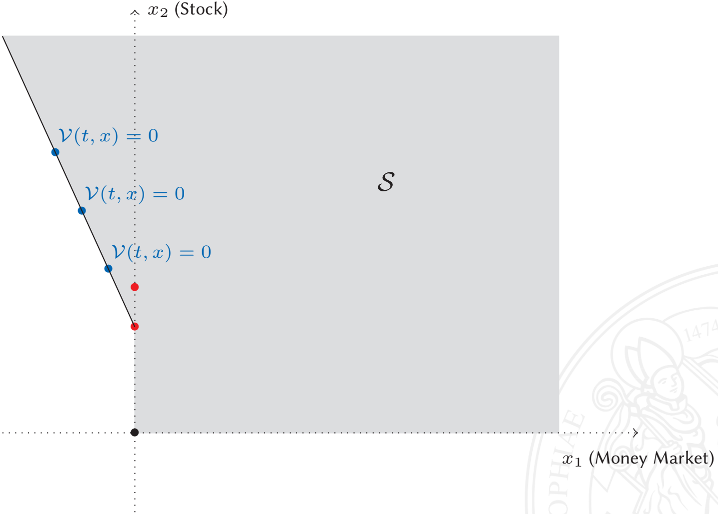
Discontinuity of the Value Function



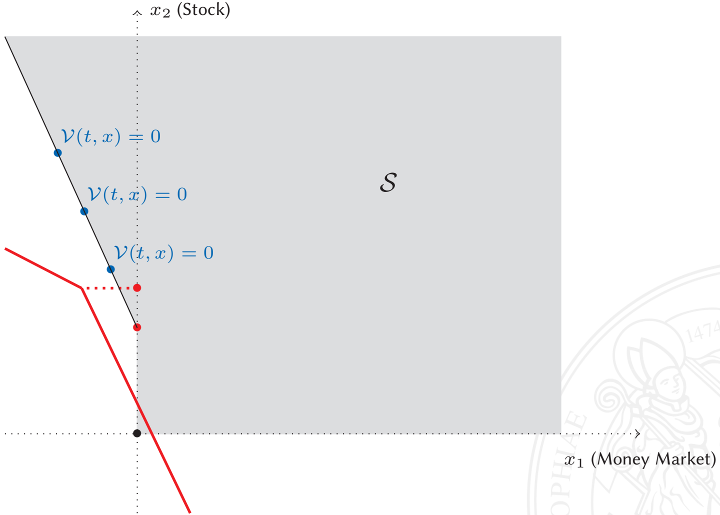
Discontinuity of the Value Function



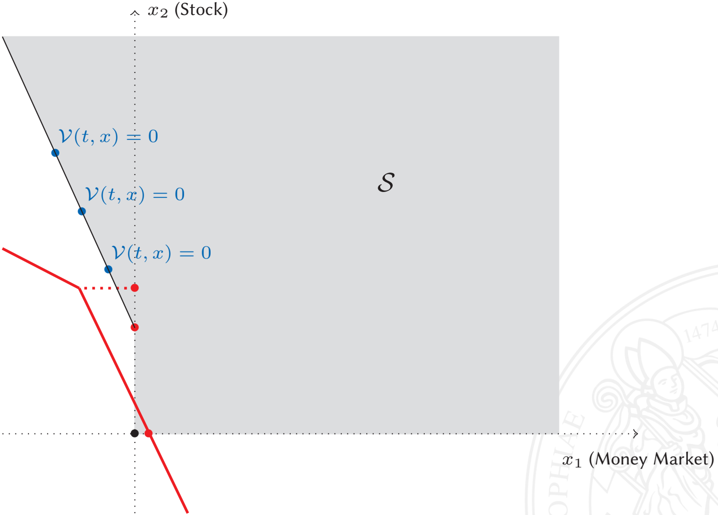
Discontinuity of the Value Function



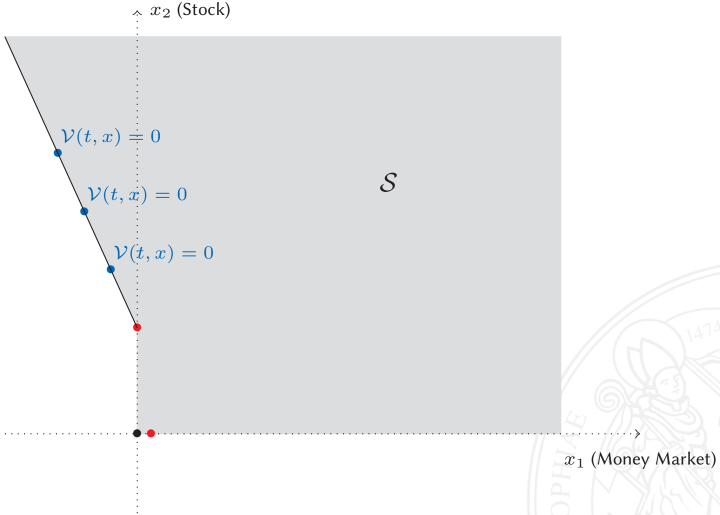
Discontinuity of the Value Function



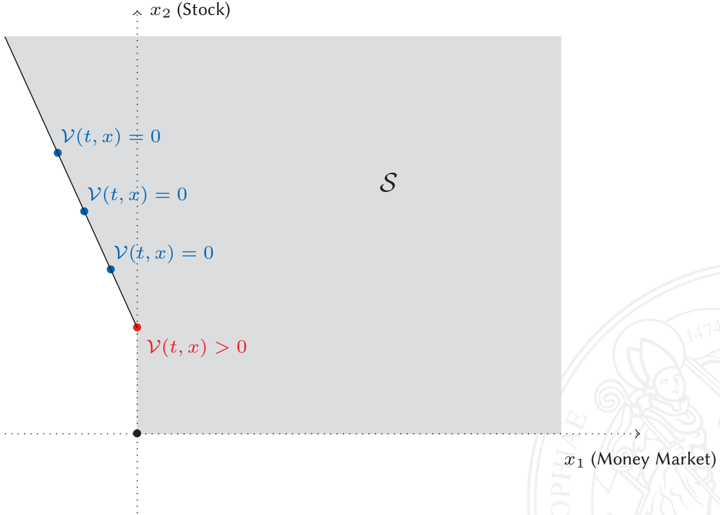
Discontinuity of the Value Function



Discontinuity of the Value Function



Discontinuity of the Value Function



Discontinuity of the Value Function

Problem: The value function is **not continuous**. In particular, the usual viscosity arguments cannot work.

Discontinuity of the Value Function

Problem: The value function is **not continuous**. In particular, the usual viscosity arguments cannot work.

But: The discontinuity is likely to appear **only** across the x_2-axis.

Discontinuity of the Value Function

Problem: The value function is **not continuous**. In particular, the usual viscosity arguments cannot work.

But: The discontinuity is likely to appear **only** across the x_2-axis.

Idea: Localize the viscosity argument by splitting the solvency region into the **borrowing** ($x_1 < 0$) and **no-borrowing** ($x_1 \geq 0$) regions.

Discontinuity of the Value Function

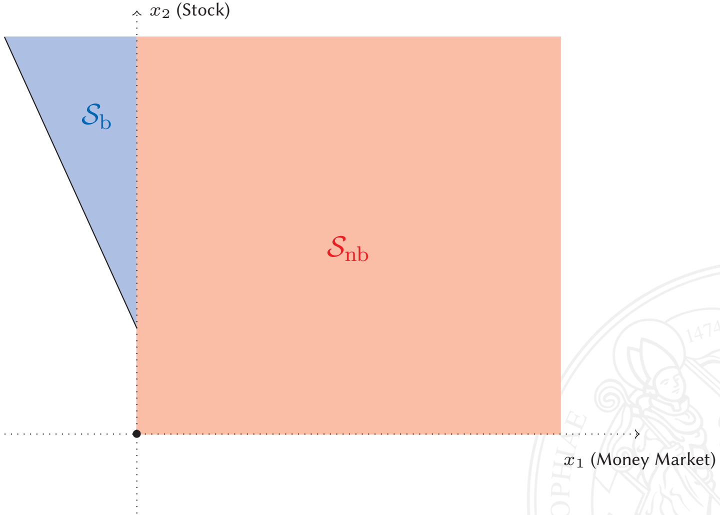
Problem: The value function is **not continuous**. In particular, the usual viscosity arguments cannot work.

But: The discontinuity is likely to appear **only** across the x_2-axis.

Idea: Localize the viscosity argument by splitting the solvency region into the **borrowing** ($x_1 < 0$) and **no-borrowing** ($x_1 \geq 0$) regions.

Difficulty: The QVIs have a **non-local** term: $\mathcal{V}(t, x) - \mathcal{M}\mathcal{V}(t, x)$.

Localizing the Solvency Region



The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity subsolution of the QVIs,

The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity subsolution of the QVIs,

(2) $v : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity supersolution of the QVIs,

The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity subsolution of the QVIs,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity supersolution of the QVIs,

(3) $u^{\mathrm{b}}(T,\cdot) \leq v_{\mathrm{b}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}}(T,\cdot) \leq v_{\mathrm{nb}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{nb}}$,

The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity subsolution of the QVIs,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity supersolution of the QVIs,

(3) $u^{\mathrm{b}}(T,\cdot) \leq v_{\mathrm{b}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}}(T,\cdot) \leq v_{\mathrm{nb}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{nb}}$,

(4) $0 = u^{\mathrm{b}} \leq v_{\mathrm{b}}$ on $[0,T] \times \overline{\partial\mathcal{S}_{\mathrm{b}}}$ and $0 = u^{\mathrm{nb}} \leq v_{\mathrm{nb}}$ on $[0,T] \times \overline{\partial\mathcal{S}_{\mathrm{nb}}}$,

The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity subsolution of the QVIs,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity supersolution of the QVIs,

(3) $u^{\mathrm{b}}(T,\cdot) \le v_{\mathrm{b}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}}(T,\cdot) \le v_{\mathrm{nb}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{nb}}$,

(4) $0 = u^{\mathrm{b}} \le v_{\mathrm{b}}$ on $[0,T] \times \overline{\partial\mathcal{S}_{\mathrm{b}}}$ and $0 = u^{\mathrm{nb}} \le v_{\mathrm{nb}}$ on $[0,T] \times \overline{\partial\mathcal{S}_{\mathrm{nb}}}$,

(5) (growth conditions),

The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity subsolution of the QVIs,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity supersolution of the QVIs,

(3) $u^{\mathrm{b}}(T,\cdot) \leq v_{\mathrm{b}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}}(T,\cdot) \leq v_{\mathrm{nb}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{nb}}$,

(4) $0 = u^{\mathrm{b}} \leq v_{\mathrm{b}}$ on $[0,T] \times \overline{\partial\mathcal{S}_{\mathrm{b}}}$ and $0 = u^{\mathrm{nb}} \leq v_{\mathrm{nb}}$ on $[0,T] \times \overline{\partial\mathcal{S}_{\mathrm{nb}}}$,

(5) (growth conditions),

(6) $x \mapsto v(t,x)$ is increasing (componentwise) for each $t \in [0,T]$.

The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity subsolution of the QVIs,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity supersolution of the QVIs,

(3) $u^{\mathrm{b}}(T,\cdot) \leq v_{\mathrm{b}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}}(T,\cdot) \leq v_{\mathrm{nb}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{nb}}$,

(4) $0 = u^{\mathrm{b}} \leq v_{\mathrm{b}}$ on $[0,T] \times \overline{\partial \mathcal{S}_{\mathrm{b}}}$ and $0 = u^{\mathrm{nb}} \leq v_{\mathrm{nb}}$ on $[0,T] \times \overline{\partial \mathcal{S}_{\mathrm{nb}}}$,

(5) (growth conditions),

(6) $x \mapsto v(t,x)$ is increasing (componentwise) for each $t \in [0,T]$.

Then $u^{\mathrm{b}} \leq v_{\mathrm{b}}$ on $[0,T] \times \overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}} \leq v_{\mathrm{nb}}$ on $[0,T] \times \overline{\mathcal{S}}_{\mathrm{nb}}$.

The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity subsolution of the QVIs,

(2) $v : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity supersolution of the QVIs,

(3) $u^{\mathrm{b}}(T, \cdot) \leq v_{\mathrm{b}}(T, \cdot)$ on $\overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}}(T, \cdot) \leq v_{\mathrm{nb}}(T, \cdot)$ on $\overline{\mathcal{S}}_{\mathrm{nb}}$,

(4) $0 = u^{\mathrm{b}} \leq v_{\mathrm{b}}$ on $[0, T] \times \overline{\partial \mathcal{S}_{\mathrm{b}}}$ and $0 = u^{\mathrm{nb}} \leq v_{\mathrm{nb}}$ on $[0, T] \times \overline{\partial \mathcal{S}_{\mathrm{nb}}}$,

(5) (growth conditions),

(6) $x \mapsto v(t, x)$ is increasing (componentwise) for each $t \in [0, T]$.

Then $u^{\mathrm{b}} \leq v_{\mathrm{b}}$ on $[0, T] \times \overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}} \leq v_{\mathrm{nb}}$ on $[0, T] \times \overline{\mathcal{S}}_{\mathrm{nb}}$.

The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity subsolution of the QVIs,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity supersolution of the QVIs,

(3) $u^{\mathrm{b}}(T,\cdot) \leq v_{\mathrm{b}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}}(T,\cdot) \leq v_{\mathrm{nb}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{nb}}$,

(4) $0 = u^{\mathrm{b}} \leq v_{\mathrm{b}}$ on $[0,T] \times \overline{\partial \mathcal{S}_{\mathrm{b}}}$ and $0 = u^{\mathrm{nb}} \leq v_{\mathrm{nb}}$ on $[0,T] \times \overline{\partial \mathcal{S}_{\mathrm{nb}}}$,

(5) (growth conditions),

(6) $x \mapsto v(t,x)$ is increasing (componentwise) for each $t \in [0,T]$.

Then $u^{\mathrm{b}} \leq v_{\mathrm{b}}$ on $[0,T] \times \overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}} \leq v_{\mathrm{nb}}$ on $[0,T] \times \overline{\mathcal{S}}_{\mathrm{nb}}$.

This implies that the value function $\mathcal{V}$ is

- **continuous** if restricted to $[0,T] \times \mathcal{S}_{\mathrm{b}}$,

The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity subsolution of the QVIs,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity supersolution of the QVIs,

(3) $u^{\mathrm{b}}(T,\cdot) \le v_{\mathrm{b}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}}(T,\cdot) \le v_{\mathrm{nb}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{nb}}$,

(4) $0 = u^{\mathrm{b}} \le v_{\mathrm{b}}$ on $[0,T] \times \overline{\partial\mathcal{S}_{\mathrm{b}}}$ and $0 = u^{\mathrm{nb}} \le v_{\mathrm{nb}}$ on $[0,T] \times \overline{\partial\mathcal{S}_{\mathrm{nb}}}$,

(5) (growth conditions),

(6) $x \mapsto v(t,x)$ is increasing (componentwise) for each $t \in [0,T]$.

Then $u^{\mathrm{b}} \le v_{\mathrm{b}}$ on $[0,T] \times \overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}} \le v_{\mathrm{nb}}$ on $[0,T] \times \overline{\mathcal{S}}_{\mathrm{nb}}$.

This implies that the value function $\mathcal{V}$ is

- **continuous** if restricted to $[0,T] \times \mathcal{S}_{\mathrm{b}}$,
- **continuous** if restricted to $[0,T] \times \mathcal{S}_{\mathrm{nb}}$,

The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity subsolution of the QVIs,

(2) $v : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity supersolution of the QVIs,

(3) $u^{\mathrm{b}}(T,\cdot) \leq v_{\mathrm{b}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}}(T,\cdot) \leq v_{\mathrm{nb}}(T,\cdot)$ on $\overline{\mathcal{S}}_{\mathrm{nb}}$,

(4) $0 = u^{\mathrm{b}} \leq v_{\mathrm{b}}$ on $[0,T] \times \overline{\partial\mathcal{S}_{\mathrm{b}}}$ and $0 = u^{\mathrm{nb}} \leq v_{\mathrm{nb}}$ on $[0,T] \times \overline{\partial\mathcal{S}_{\mathrm{nb}}}$,

(5) (growth conditions),

(6) $x \mapsto v(t,x)$ is increasing (componentwise) for each $t \in [0,T]$.

Then $u^{\mathrm{b}} \leq v_{\mathrm{b}}$ on $[0,T] \times \overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}} \leq v_{\mathrm{nb}}$ on $[0,T] \times \overline{\mathcal{S}}_{\mathrm{nb}}$.

This implies that the value function $\mathcal{V}$ is

- **continuous** if restricted to $[0,T] \times \mathcal{S}_{\mathrm{b}}$,
- **continuous** if restricted to $[0,T] \times \mathcal{S}_{\mathrm{nb}}$,
- **upper semicontinuous** everywhere,

The Local Comparison Principle

It turn out that the QVIs satisfy the following **local comparison principle**: Let

(1) $u : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity subsolution of the QVIs,

(2) $v : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ be a (local) viscosity supersolution of the QVIs,

(3) $u^{\mathrm{b}}(T, \cdot) \leq v_{\mathrm{b}}(T, \cdot)$ on $\overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}}(T, \cdot) \leq v_{\mathrm{nb}}(T, \cdot)$ on $\overline{\mathcal{S}}_{\mathrm{nb}}$,

(4) $0 = u^{\mathrm{b}} \leq v_{\mathrm{b}}$ on $[0, T] \times \overline{\partial \mathcal{S}_{\mathrm{b}}}$ and $0 = u^{\mathrm{nb}} \leq v_{\mathrm{nb}}$ on $[0, T] \times \overline{\partial \mathcal{S}_{\mathrm{nb}}}$,

(5) (growth conditions),

(6) $x \mapsto v(t, x)$ is increasing (componentwise) for each $t \in [0, T]$.

Then $u^{\mathrm{b}} \leq v_{\mathrm{b}}$ on $[0, T] \times \overline{\mathcal{S}}_{\mathrm{b}}$ and $u^{\mathrm{nb}} \leq v_{\mathrm{nb}}$ on $[0, T] \times \overline{\mathcal{S}}_{\mathrm{nb}}$.

This implies that the value function $\mathcal{V}$ is

- **continuous** if restricted to $[0, T] \times \mathcal{S}_{\mathrm{b}}$,
- **continuous** if restricted to $[0, T] \times \mathcal{S}_{\mathrm{nb}}$,
- **upper semicontinuous** everywhere,
- the **unique** viscosity solution of the QVIs.

Construction of Optimal Strategies

A Candidate Optimal Strategy

Recall that the **maximum operator** $\mathcal{M}$ was defined as

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta} \mathcal{V}\big(t, x_1 - \Delta - \gamma|\Delta| - C, x_2 + \Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward attainable if you start with an impulse.

A Candidate Optimal Strategy

Recall that the **maximum operator** $\mathcal{M}$ was defined as

$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta} \mathcal{V}\big(t, x_1 - \Delta - \gamma|\Delta| - C, x_2 + \Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward attainable if you start with an impulse.

By the **Bellman principle**, we expect that

$$\mathcal{V}(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[\mathcal{M}\mathcal{V}\big(\tau, \bar{X}(\tau)\big)\big]$$

where $\bar{X}$ denotes the **uncontrolled** portfolio process. This is an **implicit optimal stopping problem** with reward $\mathcal{M}\mathcal{V}$.

A Candidate Optimal Strategy

Recall that the **maximum operator** $\mathcal{M}$ was defined as

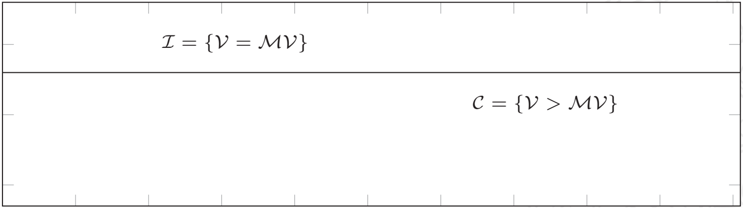
$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta} \mathcal{V}\big(t, x_1 - \Delta - \gamma|\Delta| - C, x_2 + \Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward attainable if you start with an impulse.

By the **Bellman principle**, we expect that

$$\mathcal{V}(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[\mathcal{M}\mathcal{V}\big(\tau, \bar{X}(\tau)\big)\big]$$

where $\bar{X}$ denotes the **uncontrolled** portfolio process. This is an **implicit optimal stopping problem** with reward $\mathcal{M}\mathcal{V}$.



A Candidate Optimal Strategy

Recall that the **maximum operator** $\mathcal{M}$ was defined as

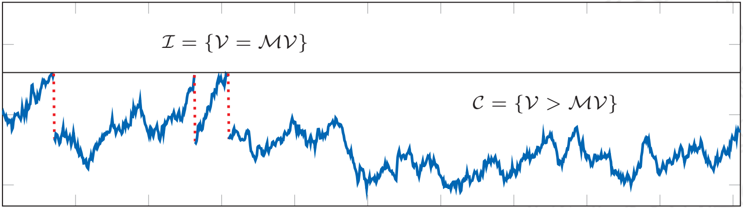
$$\mathcal{M}\mathcal{V}(t,x) \triangleq \sup_{\Delta} \mathcal{V}\big(t, x_1 - \Delta - \gamma|\Delta| - C, x_2 + \Delta)\big).$$

Interpretation: $\mathcal{M}\mathcal{V}$ highest reward attainable if you start with an impulse.

By the **Bellman principle**, we expect that

$$\mathcal{V}(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\big[\mathcal{M}\mathcal{V}\big(\tau, \bar{X}(\tau)\big)\big]$$

where $\bar{X}$ denotes the **uncontrolled** portfolio process. This is an **implicit optimal stopping problem** with reward $\mathcal{M}\mathcal{V}$.



The Verification Theorem

Let $\mathbb{H}$ be the set of upper semicontinuous functions $h : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ with

The Verification Theorem

Let $\mathbb{H}$ be the set of upper semicontinuous functions $h : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ with

(H1) h is **superharmonic** with respect to the uncontrolled state process, i.e. $h(\cdot, \bar{X})$ is a strong supermartingale,

The Verification Theorem

Let $\mathbb{H}$ be the set of upper semicontinuous functions $h : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ with

(H1) h is **superharmonic** with respect to the uncontrolled state process, i.e. $h(\cdot, \bar{X})$ is a strong supermartingale,

(H2) h **dominates the reward**, i.e. $h \geq \mathcal{M}h$,

The Verification Theorem

Let $\mathbb{H}$ be the set of upper semicontinuous functions $h : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ with

- **(H1)** h is **superharmonic** with respect to the uncontrolled state process, i.e. $h(\cdot, \bar{X})$ is a strong supermartingale,
- **(H2)** h **dominates the reward**, i.e. $h \geq \mathcal{M}h$,
- **(H3)** $h = \mathcal{V}$ on the boundary of the state space.

The Verification Theorem

Let $\mathbb{H}$ be the set of upper semicontinuous functions $h : [0, T] \times \overline{\mathcal{S}} \to \mathbb{R}$ with

(H1) h is **superharmonic** with respect to the uncontrolled state process, i.e. $h(\cdot, \bar{X})$ is a strong supermartingale,

(H2) h **dominates the reward**, i.e. $h \geq \mathcal{M}h$,

(H3) $h = \mathcal{V}$ on the boundary of the state space.

Verification Theorem

$\mathcal{V} \in \mathbb{H}$ is the pointwise minimum of $\mathbb{H}$ and the candidate optimal strategy defined in terms of $\{\mathcal{V} = \mathcal{M}\mathcal{V}\}$ and $\{\mathcal{V} > \mathcal{M}\mathcal{V}\}$ is indeed optimal.

The Verification Theorem

Let $\mathbb{H}$ be the set of upper semicontinuous functions $h : [0,T] \times \overline{\mathcal{S}} \to \mathbb{R}$ with

(H1) h is **superharmonic** with respect to the uncontrolled state process, i.e. $h(\cdot, \bar{X})$ is a strong supermartingale,

(H2) h **dominates the reward**, i.e. $h \geq \mathcal{M}h$,

(H3) $h = \mathcal{V}$ on the boundary of the state space.

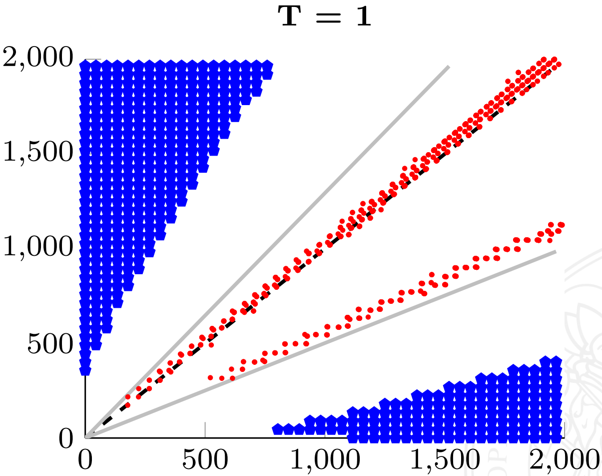
Verification Theorem

$\mathcal{V} \in \mathbb{H}$ is the pointwise minimum of $\mathbb{H}$ and the candidate optimal strategy defined in terms of $\{\mathcal{V} = \mathcal{M}\mathcal{V}\}$ and $\{\mathcal{V} > \mathcal{M}\mathcal{V}\}$ is indeed optimal.

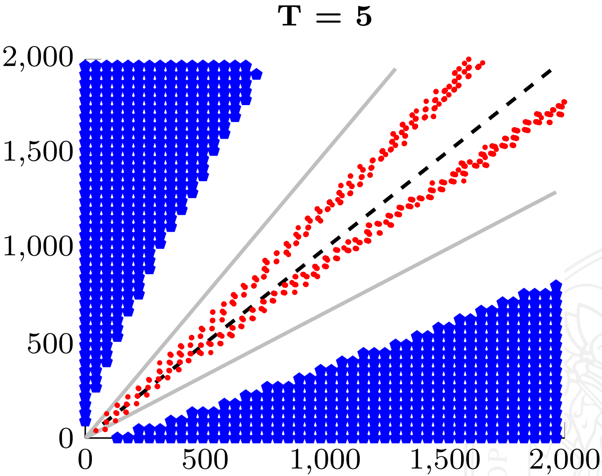
Proof: Iteratively solve the implicit optimal stopping problem. Uses classical optimal stopping techniques and that $\mathcal{V}$ is the pointwise minimum of $\mathbb{H}$.

Numerical Results

Numerical Example: Power Utility



Numerical Example: Power Utility



Conclusion

Conclusions:

- We solve a **intractable** utility maximization problem involving constant costs.

Conclusion

Conclusions:

- We solve a **intractable** utility maximization problem involving constant costs.
- The value function is **discontinuous**, even in the **interior** of the state space.

Conclusion

Conclusions:

- We solve a **intractable** utility maximization problem involving constant costs.
- The value function is **discontinuous**, even in the **interior** of the state space.
- We derive a **uniqueness** result for a discontinuous viscosity solution which implies **piecewise continuity**.

Conclusion

Conclusions:

- We solve a **intractable** utility maximization problem involving constant costs.
- The value function is **discontinuous**, even in the **interior** of the state space.
- We derive a **uniqueness** result for a discontinuous viscosity solution which implies **piecewise continuity**.
- To the best of our knowledge, this is the first **local** viscosity comparison principle for a **non-local PDE**. Put differently, this is the first uniqueness result for purely discontinuous viscosity solutions, which still yields some **regularity**.

Conclusion

Conclusions:

- We solve a **intractable** utility maximization problem involving constant costs.
- The value function is **discontinuous**, even in the **interior** of the state space.
- We derive a **uniqueness** result for a discontinuous viscosity solution which implies **piecewise continuity**.
- To the best of our knowledge, this is the first **local** viscosity comparison principle for a **non-local PDE**. Put differently, this is the first uniqueness result for purely discontinuous viscosity solutions, which still yields some **regularity**.
- We construct **optimizers** for a discontinuous control problem.

Conclusions:

- We solve a **intractable** utility maximization problem involving constant costs.
- The value function is **discontinuous**, even in the **interior** of the state space.
- We derive a **uniqueness** result for a discontinuous viscosity solution which implies **piecewise continuity**.
- To the best of our knowledge, this is the first **local** viscosity comparison principle for a **non-local PDE**. Put differently, this is the first uniqueness result for purely discontinuous viscosity solutions, which still yields some **regularity**.
- We construct **optimizers** for a discontinuous control problem.

Thanks for Listening!

Papers:

▷ Belak, C., Christensen, S., and Seifried, F. T.: **A general verification result for stochastic impulse control problems**, *SIAM Journal on Control and Optimization*, Vol. 55, No. 2, pp. 627-649, 2017.

▷ Belak, C. and Christensen, S.: **Utility maximization in a factor model with constant and proportional costs**, Preprint, available on SSRN, 2017.