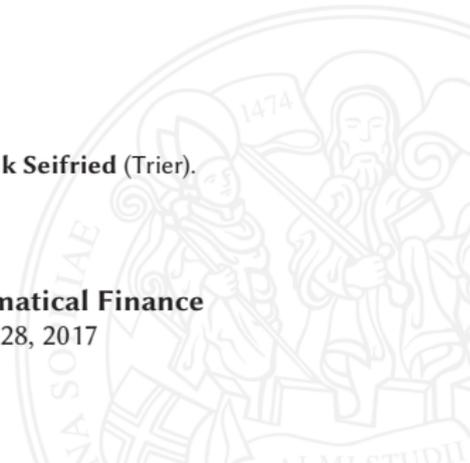# Universität Trier

Non-Smooth Verification for Impulse Control Problems

Christoph Belak

Department IV – Mathematics
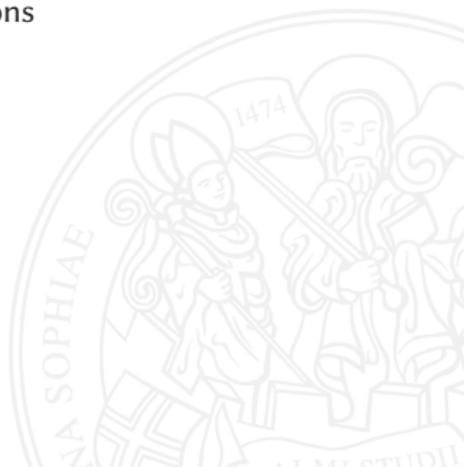University of Trier
Germany

Joint work with **Sören Christensen** (Hamburg) and **Frank Seifried** (Trier).

Byrne Young Researcher Workshop on Mathematical Finance
University of Michigan, Ann Arbor, March 28, 2017

Outline

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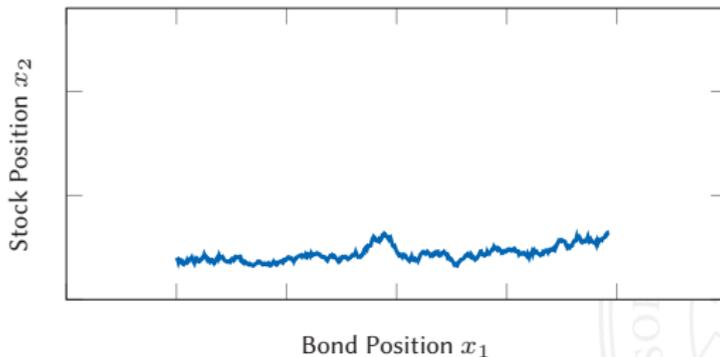
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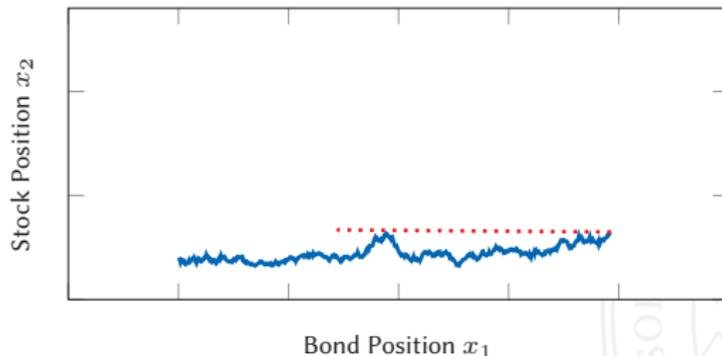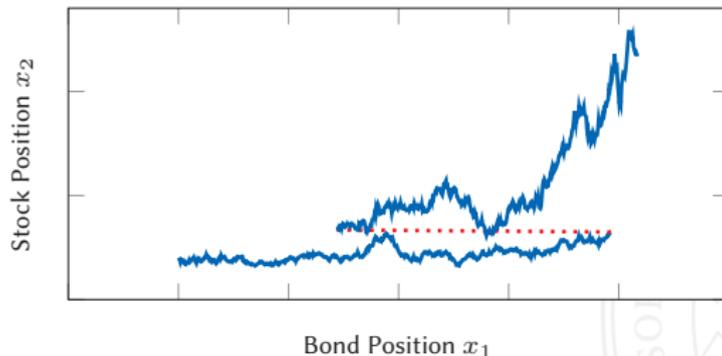- Δ_k denotes the **volume** of the kth transaction (random variable).

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- **Classical verification**: Eastham and Hastings (1988), Korn (1998), Bielecki and Pliska (2000), Liu (2004).
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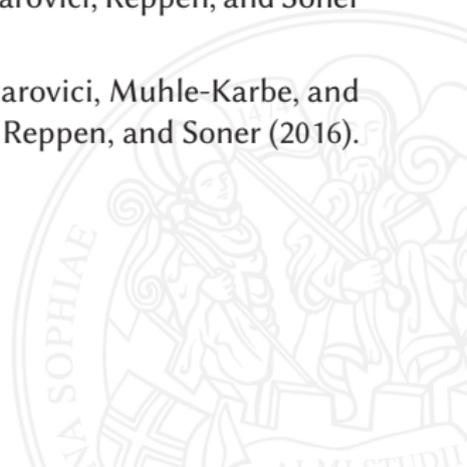
Related Literature

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To the best of our knowledge, existence of optimal controls is **unknown** — even in a Black-Scholes setting with $\log$- or power utility.

The General Impulse Control Problem

Consider an $\mathbb{R}^n$-valued **system** $X = X^\Lambda$ controlled by an **impulse control** $\Lambda = \{(\tau_k, \Delta_k)\}_{k \in \mathbb{N}}$ as follows:

$$dX(t) = \mu(X(t))dt + \sigma(X(t))\, dW(t), \qquad t \in [\tau_k, \tau_{k+1}),$$
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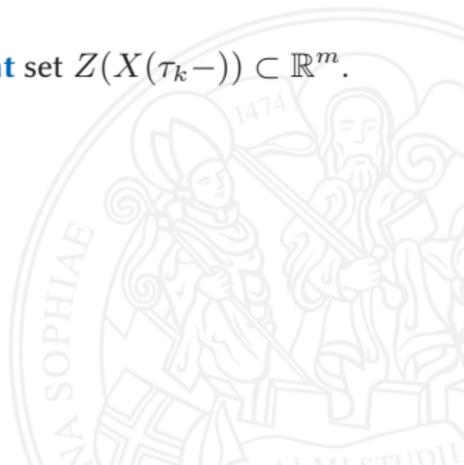
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The **objective** is to find a maximizer of

$$\mathcal{V}(t,x) = \sup_{\Lambda \in \mathcal{A}(t,x)} \mathbb{E}\Big[\sum_{k \in \mathbb{N}} K(X_{t,x}^\Lambda(\tau_k-), \Delta_k)\mathbb{1}_{\{\tau_k \le T\}} + g(X_{t,x}^\Lambda(T))\Big].$$

The martingale optimality principle of stochastic control suggests that the value function $\mathcal{V}$ can be linked to the **quasi-variational inequalities** (QVIs)

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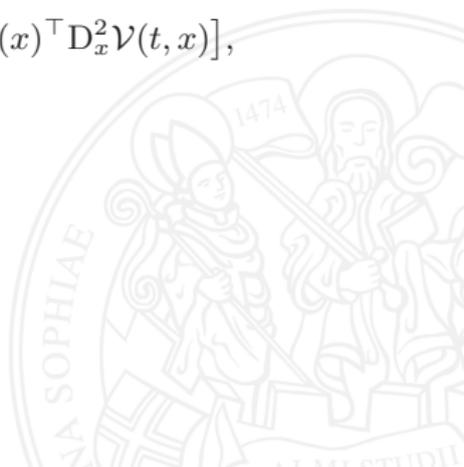
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A Candidate Optimal Control

Observe that

- if $\mathcal{V}(t, x) > \mathcal{MV}(t, x)$, an impulse in state (t, x) **cannot be optimal**, and
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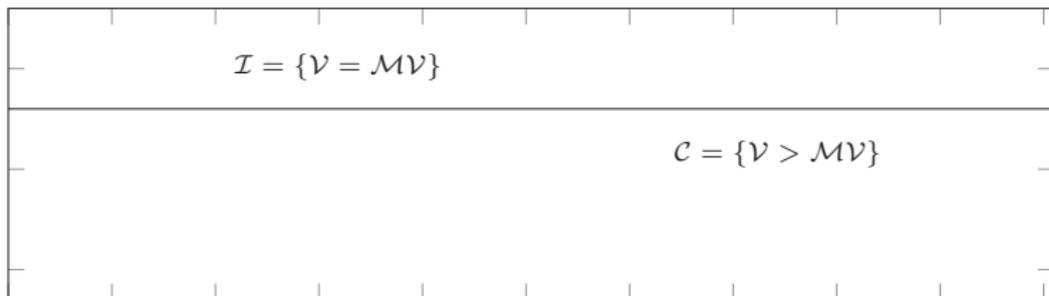$\mathcal{C} = \{\mathcal{V} > \mathcal{MV}\}$

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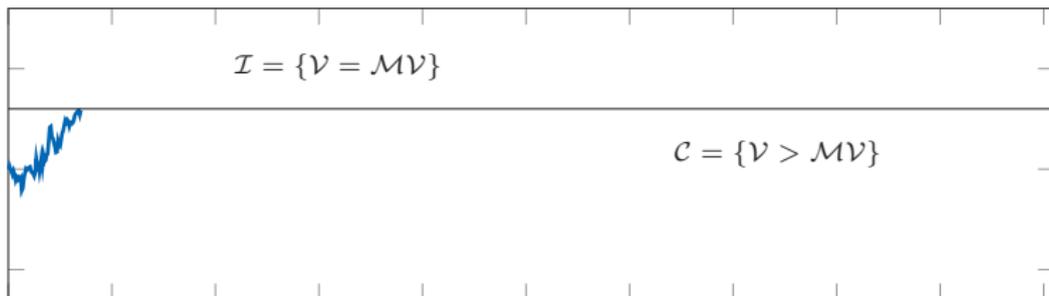
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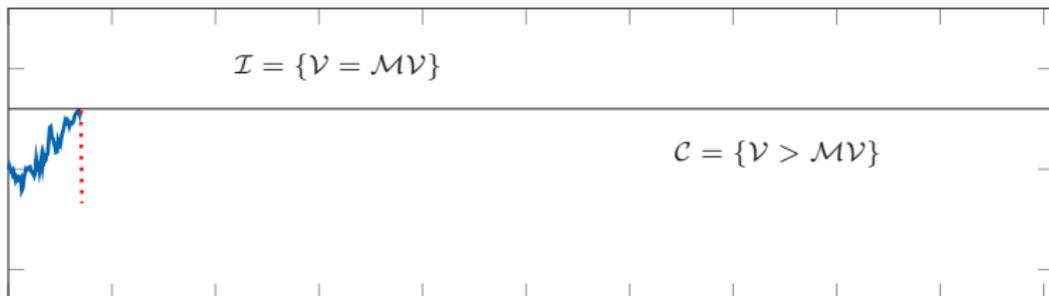
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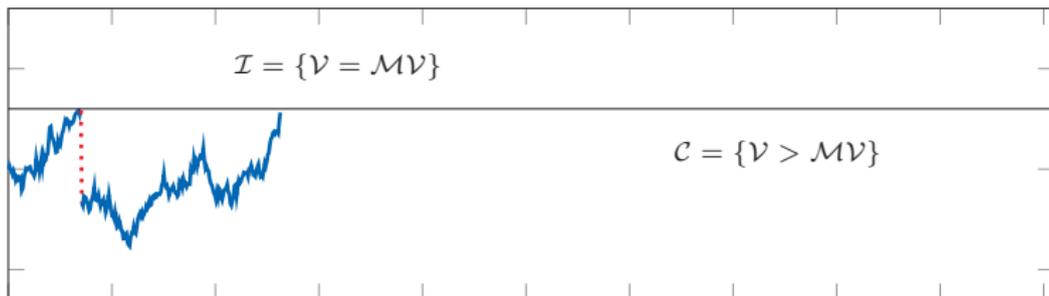
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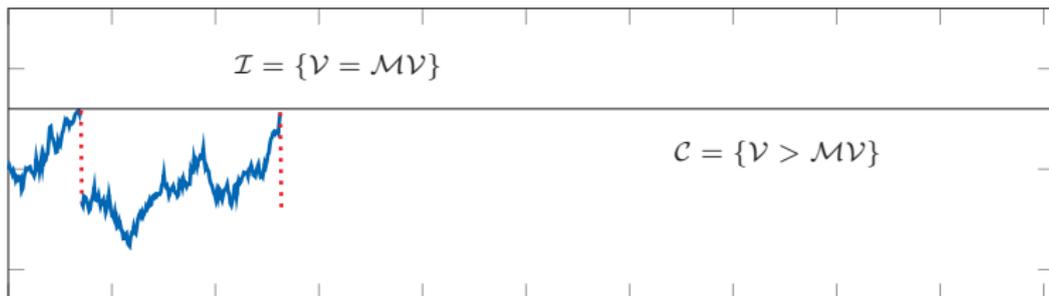
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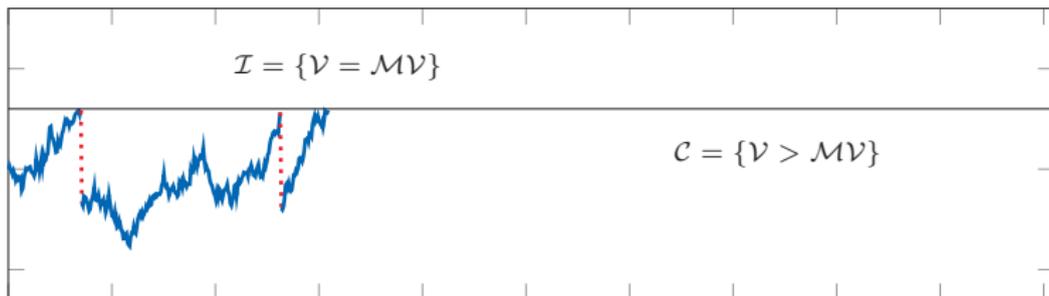
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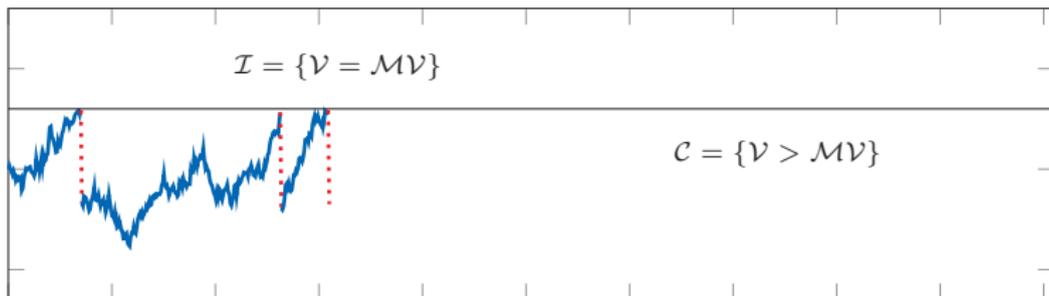# A Candidate Optimal Control

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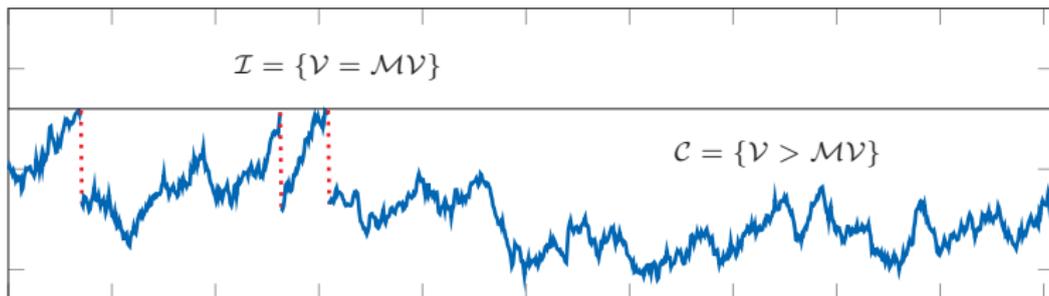- if $\mathcal{V}(t, x) > \mathcal{MV}(t, x)$, an impulse in state (t, x) **cannot be optimal**, and
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Problem: Verification requires a solution of the QVIs which is sufficiently smooth to apply Itô's formula.

Non-Smooth Verification via Superharmonic Functions

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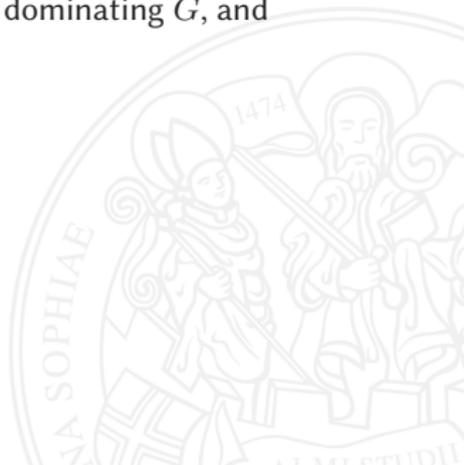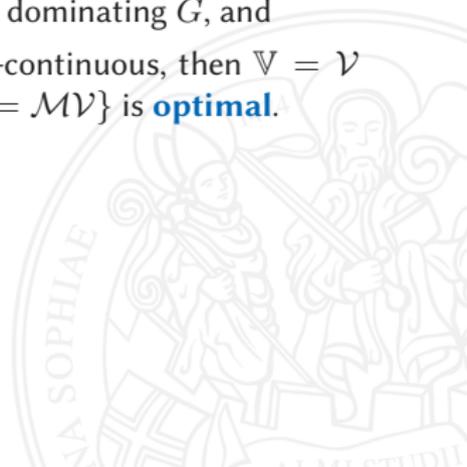
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Remark: Under standard assumptions, $G = \mathcal{M}\mathbb{V}$ is upper semi-continuous if $\mathbb{V}$ is upper semi-continuous. That is, to verify optimality, continuity of $\mathbb{V}$ should suffice!

Let $\mathbb{H}$ be the set of **upper semi-continuous** functions $h : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ with

(H1) h is **superharmonic** with respect to the uncontrolled state process,

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Note: If $\mathcal{V}$ is known to be continuous, then $\mathcal{V} \in \mathbb{H}$ and things become easy.

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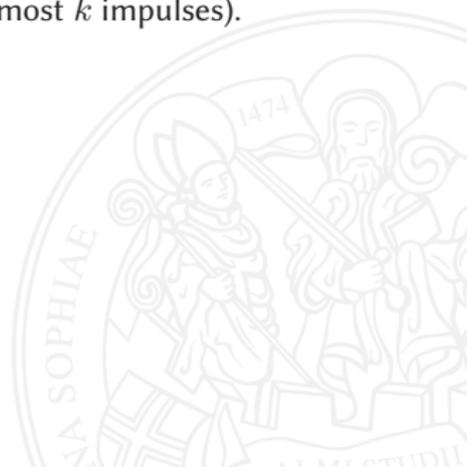
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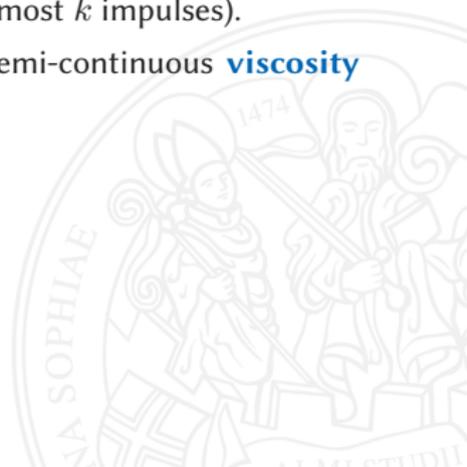
(3) Show that the limit $\mathfrak{V} \triangleq \lim_{k\to\infty} v_k$ is a lower semi-continuous **viscosity supersolution** of the QVIs.

(4) Then $\mathfrak{V} \leq \mathcal{V} \leq \mathbb{V}$. Now apply viscosity comparison so that $\mathfrak{V} \geq \mathbb{V}$ and hence

$$\mathcal{V} = \mathbb{V} = \mathfrak{V}$$

is **continuous** and the **unique viscosity solution** of the QVIs.

Sufficient Conditions

Our procedure is based on **three ingredients**:

- Superharmonic function techniques in optimal stopping
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- The candidate optimal control is admissible
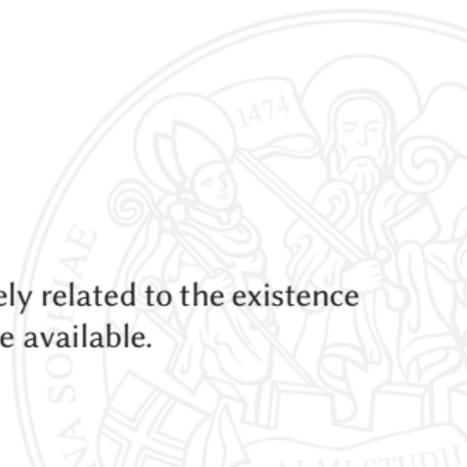
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Note: The existence of Ψ and viscosity comparison is closely related to the existence of a strict supersolution of the QVIs. Candidates for Ψ are available.

Some additional remarks:

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- Proving **admissibility** can be difficult and has to be done case-by-case.
- **Advantage** over the classical approach: The candidate optimal control is defined in terms of the value function, making it easier to verify admissibility (at least in principle).

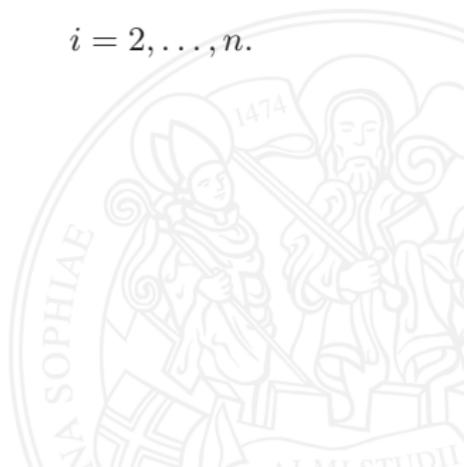# Solving the Transaction Cost Problem

The Market Model

We assume that the portfolio $X = \{X(t)\}_{t \in [0,T]}$ evolves as

$$
\begin{aligned}
\mathrm{d}Y(t) &= \alpha(Y(t))\mathrm{d}t + \beta(Y(t))\mathrm{d}W(t), & t &\in [0,T], \\
\mathrm{d}X(t) &= \mathrm{diag}(X(t))\big[\mu(Y(t))\mathrm{d}t + \sigma(Y(t))\mathrm{d}W(t)\big], & t &\in [\tau^k, \tau^{k+1}), \\
X(\tau^k) &= \Gamma(X(\tau^k-), \Delta^k),
\end{aligned}
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where $\Gamma(x, \Delta) = \xi$ with

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\xi_1 &= x_1 - \sum_{j=2}^{n}[\Delta_j + \gamma_j|\Delta_j| + K_j]\mathbb{1}_{\{\Delta_j \neq 0\}}, \\
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A portfolio $x \in \mathbb{R}^n$ is **solvent** if it has a **positive liquidation** value $\mathrm{L}(x)$, i.e.,

$$\mathrm{L}(x) \triangleq x_1 + \sum_{j=2}^{n}[x_j - \gamma_j|x_j| - K_j] > 0.$$

The set $\mathcal{S} \subset \mathbb{R}^n$ of solvent portfolios is called the **solvency region**.

Now fix a **utility function** $U : \mathbb{R}_+ \to \mathbb{R}$ such that

- U is strictly increasing and continuous (not necessarily concave),
- U is lower bounded; without loss of generality $U(0) = 0$,
- U satisfies $U(l) \leq C(1 + |l|^p)$ for some $C > 0$, $p \in (0, 1)$.

The Optimization Criterion

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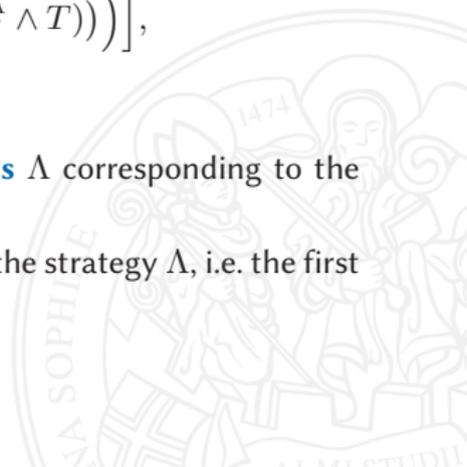
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The objective is to **maximize utility of terminal wealth**, i.e.

$$\mathcal{V}(t, x, y) = \sup_{\Lambda \in \mathcal{A}(t,x,y)} \mathbb{E}\Big[U\Big(\mathrm{L}\big(X_{t,x,y}^{\Lambda}(\tau_{\mathcal{S}}^{\Lambda} \wedge T))\big)\Big)\Big],$$

where

- $\mathcal{A}(t, x, y)$ denotes the set of **admissible strategies** Λ corresponding to the initial state (t, x, y).
- $\tau_{\mathcal{S}}^{\Lambda}$ denotes the **bankruptcy time** corresponding to the strategy Λ, i.e. the first exit time of $X_{t,x,y}^{\Lambda}$ from the solvency region $\mathcal{S}$.

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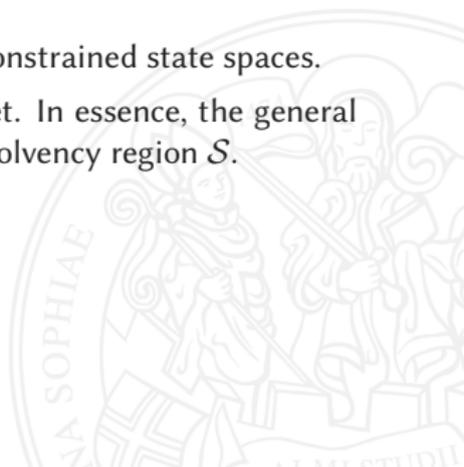
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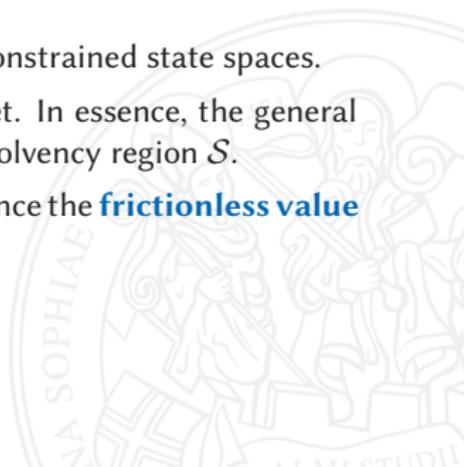
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- Admissibility of the candidate optimal strategy can be verified. **Idea**: The investor goes bankrupt before the trading times accumulate.

Conclusion and Outlook

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- The results open the door to a lot of interesting questions...
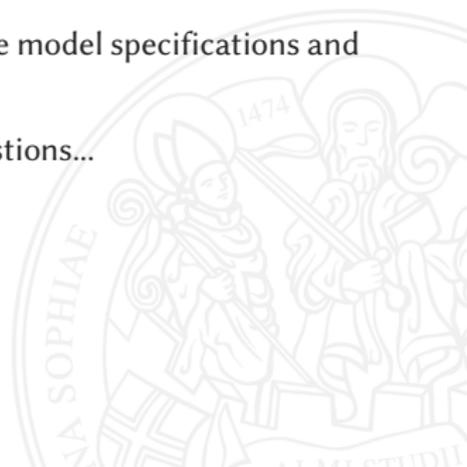
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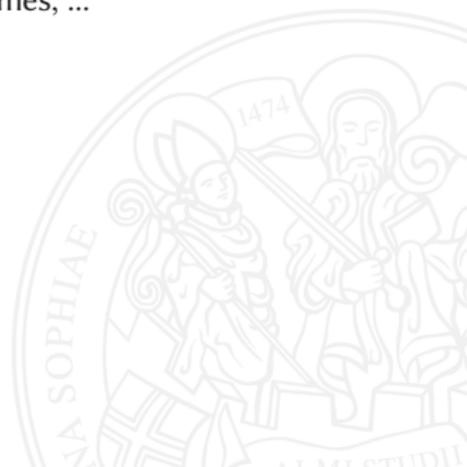- Devise a **numerical scheme** to compute the trading regions.

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Thanks for your attention!