Universität Trier

# Backward Nonlinear Expectation Equations

**Christoph Belak**

Department IV – Mathematics
University of Trier
Germany

Joint work with **Thomas Seiferling** and **Frank Seifried**.

**Colloquium on Mathematical Statistics and Stochastic Processes**
University of Hamburg, November 29, 2016

# Outline

**(1)** Recursive and Stochastic Differential Utility

**(2)** Nonlinear Expectations and Model Uncertainty

**(3)** Backward Nonlinear Expectation Equations

**(4)** Application to Recursive Utility under Model Uncertainty

# Recursive and Stochastic Differential Utility

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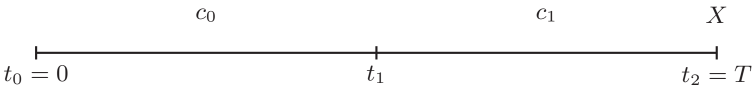

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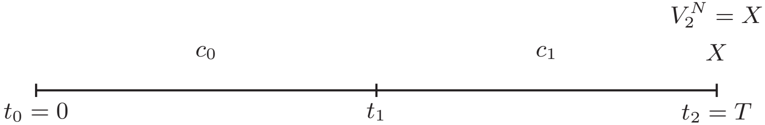

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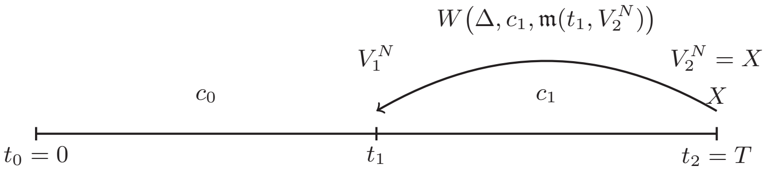

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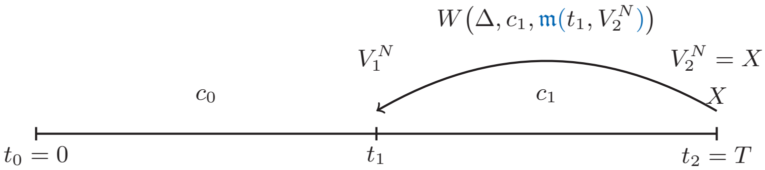


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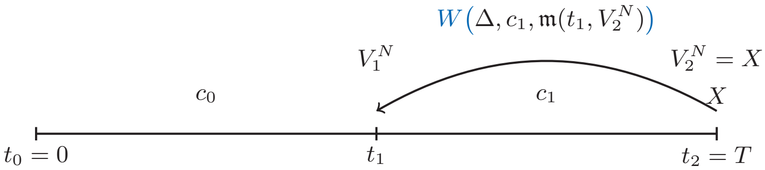


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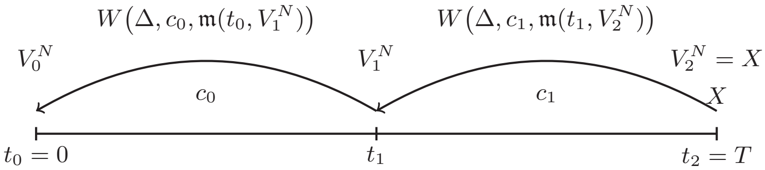


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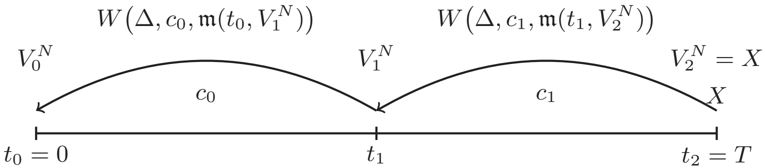


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$$\overset{\Delta=1}{\Rightarrow} \mathfrak{v}_N(c, X) \triangleq V_0^N = \ldots = U^{-1}\left(\mathbb{E}\left[\sum_{n=0}^{N-1} e^{-rt_n} U(c_n) + e^{-rT} U(X)\right]\right).$$

# Stochastic Differential Utility (Duffie/Epstein 1992)

In a continuous-time setting, **stochastic differential utility** is defined axiomatically through

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where $V = (V_t)_{t \in [0,T]}$ is given as the solution of the **backward stochastic differential equation** (BSDE)

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**Example**: If $f(c, v) = U(c) - rv$, then

$$\mathfrak{v}(c, X) = V_0 = \mathbb{E} \left[ \int_0^T e^{-rt} U(c_t)\, \mathrm{d}t + e^{-rT} U(X) \right].$$

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**Kraft/Seifried (2014)**: Suppose that

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In particular, $\mathfrak{v}_N(c, X) = V_0^N \to V_0 = \mathfrak{v}(c, X)$ and

$$f(c, v) = \frac{\partial}{\partial \Delta} W(\Delta, c, v)|_{\Delta=0}.$$

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**Research Question 2**

Does the discrete-time recursive utility index still converge, if we replace the conditional expectation $\mathbb{E}_t$ with a more general certainty equivalent $\mathfrak{m}$?

# Nonlinear Expectations and Model Uncertainty

# Research Questions

## Model Uncertainty

Aim: Generalize the results to certainty equivalents of the form

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If the sets $\mathcal{Q}_t$ are chosen carefully, the operator $\mathcal{E}_t$ satisfies the same properties as the conditional expectation, **except for linearity**.

**Guess**: The continuous-time object is of the form

$$V_t = \mathcal{E}_t\left[\int_t^T f(c_s, V_s)\mathrm{d}s + U(X)\right], \qquad t \in [0, T].$$

We call this a **backward nonlinear expectation equation** (BNEE).

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**Examples**: drift uncertainty (g-Expectations), volatility uncertainty ((random) G-Expectations), dynamic risk measures, ...

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**Notice**: There is no $\sigma$-field and no probability measure involved here!

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# Backward Nonlinear Expectation Equations

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**Remark**: $g$-Expectations, $G$-Expectations and **random $G$-Expectations** are regular nonlinear expectations!

# Existence and Uniqueness of BNEEs

Let $\xi \in L_T$ and consider the **general BNEE**

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**Existence and Uniqueness of BNEEs**

If $\mathcal{E}_t$ is **regular**, then the BNEE admits a unique solution in $D$.

# Discretization of BNEEs

Let $\mathcal{E}_t$ be a **regular** nonlinear expectation and consider the BNEE

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$$X_t = \mathcal{E}_t\left[\int_t^T g(s, X_s)\,\mathrm{d}s + \xi\right], \qquad t \in [0,T],$$

as well as the **discrete-time backward aggregation** $X_N^N \triangleq \xi$,

$$X_k^N \triangleq \mathcal{E}_{t_k^N}\left[g\big(t_k^N, \mathcal{E}_{t_k^N}[X_{k+1}^N]\big)\Delta_{k+1}^N + X_{k+1}^N\right], \qquad k = N-1, \ldots, 0,$$

for a partition $\Delta^N : 0 = t_0^N < \ldots < t_N^N = T$, $\Delta_k^N \triangleq t_k^N - t_{k-1}^N$. Assume that $(\Delta^N)_{N\in\mathbb{N}}$ is **infinitesimal**.

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**Discrete-Time Approximation of BNEEs**

If the mapping $t \mapsto g(t, \eta)$ is left-continuous for every $\eta \in L_T$, then

$$\max_{k=0,\ldots,N} \|X_k^N - X_{t_k^N}\|_L \to 0 \qquad \text{as } N \to \infty.$$

# Recursive Utility under Model Uncertainty

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**(iii)** $c_t^N \to c_t$ for $\mathrm{d}t$-almost every $t \in [0, T]$.

We fix an **intertemporal aggregator**

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**(v)** $f(0, c, v) \in L_t$ whenever $c, v \in L_t$.

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## Recursive Utility Process

The **recursive utility process** $V^N = \{V_k^N\}_{k\in\{0,\dots,N\}}$ is defined as

$$V_N^N \triangleq \xi,$$
$$V_k^N \triangleq W\left(\Delta_k^N, c_{t_k^N}^N, \mathcal{E}_{t_k^N}\left[V_{k+1}^N\right]\right) \qquad \text{for all } k = N-1, \dots, 0.$$

# Recursive and Stochastic Differential Utility under Uncertainty

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## Stochastic Differential Utility Process

The **stochastic differential utility process** $V = \{V_t\}_{t\in[0,T]}$ is defined as the unique solution of the BNEE

$$V_t \triangleq \mathcal{E}_t\left[\int_t^T f(0, c_s, V_s)\mathrm{d}s + \xi\right] \qquad \text{for all } t \in [0,T].$$

# The Convergence Result

## Theorem

The recursive utility processes $V^N$ converge to the stochastic differential utility process $V$ in the sense that

$$\max_{k=0,\dots,N} \left\| V_k^N - V_{t_k^N} \right\|_L \to 0 \qquad \text{as } N \to \infty.$$

In particular, the utility indices converge:

$$\mathfrak{v}_N(c^N, \xi) \triangleq V_0^N = V_0 \triangleq \mathfrak{v}(c, \xi).$$

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**(4)** We study existence, uniqueness, stability and discrete-time approximation of BNEEs.

**(5)** We justify the axiomatic definition of stochastic differential utility under nonlinear expectations by identifying the utility process as the **continuous-time limit** of the discrete-time recursive utility process.

**Thanks for your attention!**

Belak, Seiferling, and Seifried (2016):
**Backward Nonlinear Expectation Equations**

Available at: **www.belak.ch/publications/**