

Pricing Contingent Claims under Jump Uncertainty

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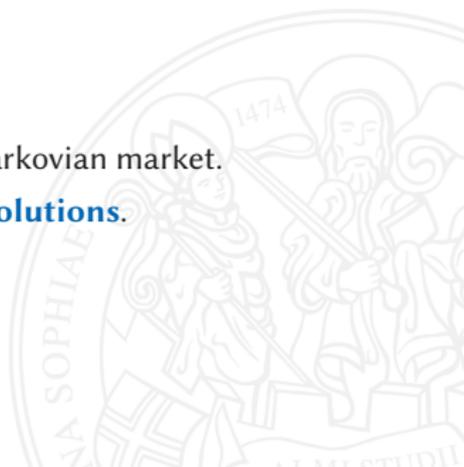
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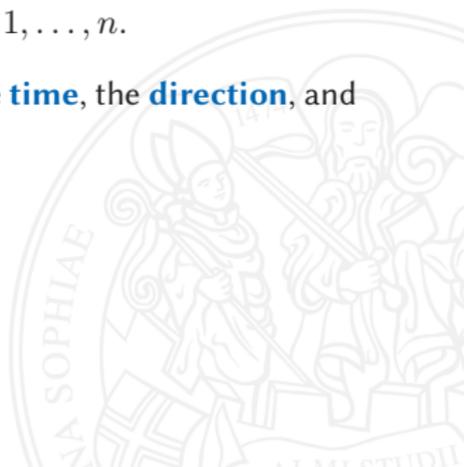
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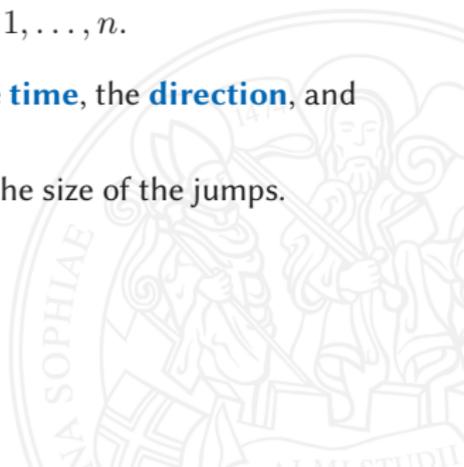
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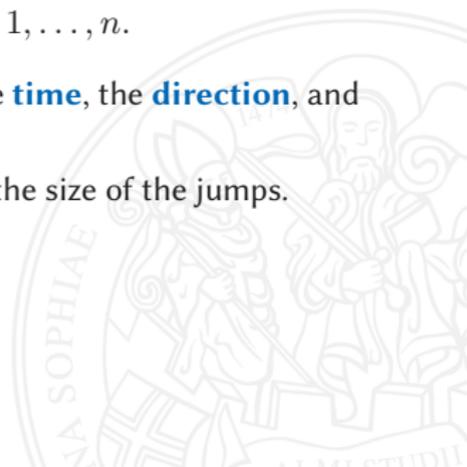
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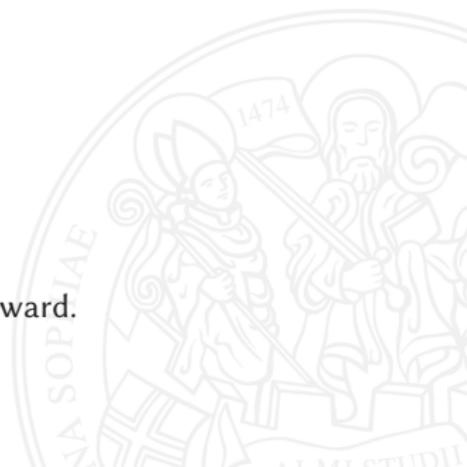
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Literature: Hua/Wilmott (1997), Mönnig (2012).



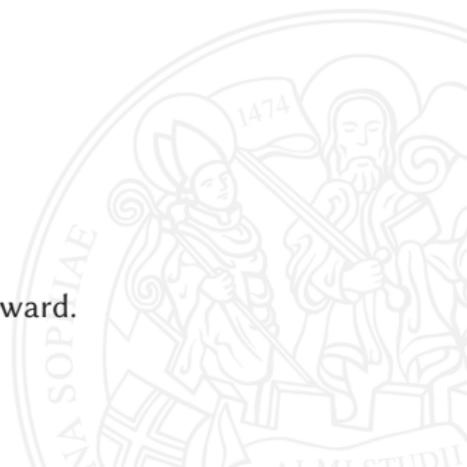
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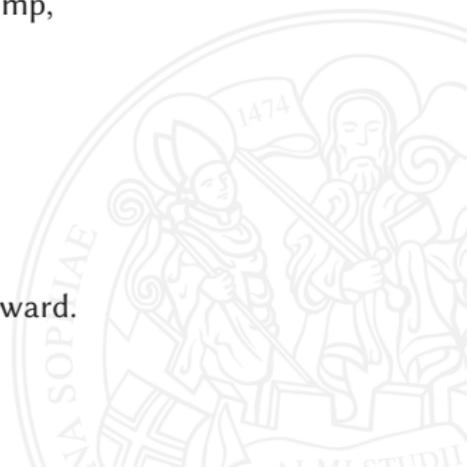
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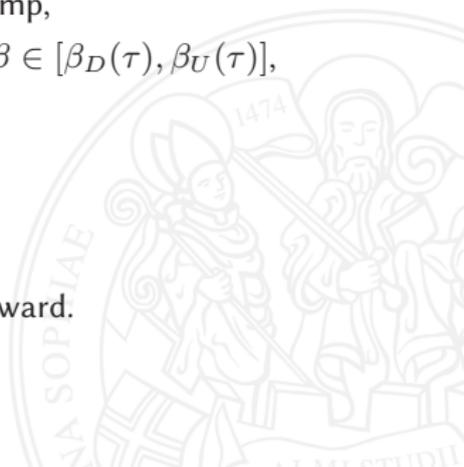
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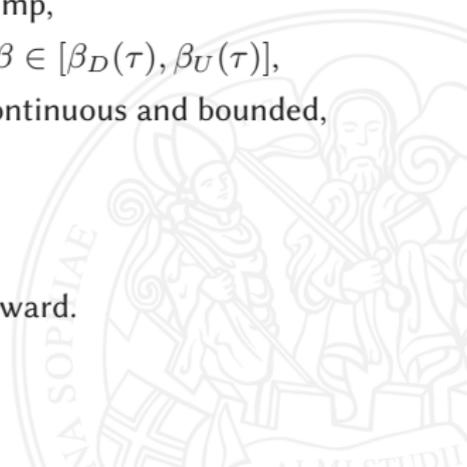
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The Jump Uncertainty Price

A **contingent claim** $\xi(P^0(T), P(T))$ is an $\mathcal{F}(T)$ -measurable, non-negative random variable, Lipschitz continuous in the prices (P^0, P) of the underlyings.



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Definition: Jump Uncertainty Price

The **jump uncertainty price** \mathcal{V}_1 is defined as

$$\mathcal{V}_1 \triangleq \inf \left\{ x \geq 0 : \exists (\zeta_1, \zeta_0) \text{ s.t.} \right. \\ \left. X_{t,x}^{\zeta_1, \zeta_0, \vartheta}(T) \geq \xi(P^0(T), P^\vartheta(T)) \text{ for every jump } \vartheta \right\}.$$

In other words, \mathcal{V}_1 is the smallest initial wealth that is required to superhedge the claim in **every jump scenario**.

The Main Idea

In order to superhedge the claim, the trader has to ensure that the wealth after a jump dominates the price in the jump-free market.



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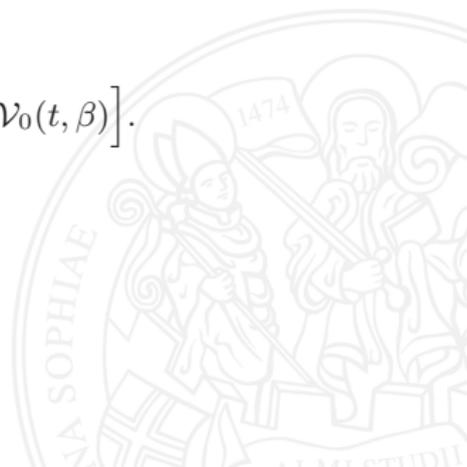
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Mathematically, this means that

$$H(\tau, X_x^{\zeta_1}(\tau-), \zeta_1(\tau)) \geq 0 \quad \text{for all stopping times } \tau,$$

where the **jump constraint** H is defined as

$$H(t, x, \zeta) \triangleq \inf_{\beta \in [\beta_D(t), \beta_U(t)]} [x + \zeta^\top \beta - \mathcal{V}_0(t, \beta)].$$



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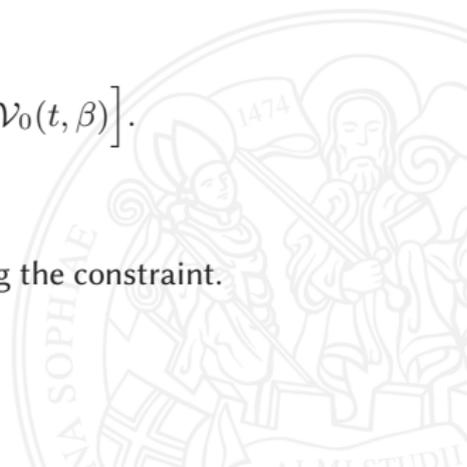
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\mathcal{V}_1 is **well-defined** if and only if there are x, ζ_1 satisfying the constraint.



Theorem (BSDE Characterization of \mathcal{V}_1)

The jump uncertainty price is given by $\mathcal{V}_1 = X(0)$, where (X, ζ_1) denotes the **minimal supersolution** of the BSDE

$$\begin{aligned}dX(t) &= [r(t)X(t) + \zeta_1(t)^\top \sigma(t)\theta(t)] dt + \zeta_1(t)^\top \sigma(t) dW(t), \\ X(T) &\geq \xi(P^0(T), P(T)).\end{aligned}$$

under the **constraint**

$$H(t, X(t), \zeta_1(t)) \geq 0 \quad \text{for all } t \in [0, T], \mathbb{P}\text{-a.s.}$$

Literature: Peng (1999), Kharroubi/Ma/Pham/Zhang (2010).

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In the case of **Markovian price dynamics**, $\mathcal{V}_1 = \mathcal{V}_1(t, p_0, p)$

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Remark: \mathcal{L} is the generator of (P^0, P) under the risk neutral measure.

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satisfying the **terminal condition**

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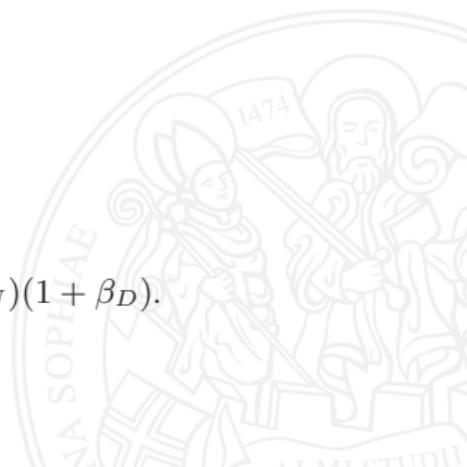
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The terminal condition is given explicitly as

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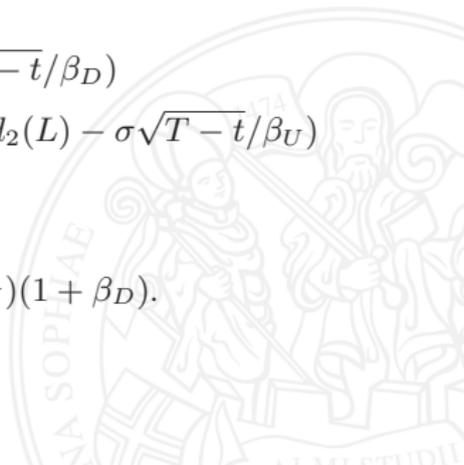
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The jump uncertainty price \mathcal{V}_1 is given **explicitly** as

$$\begin{aligned} \mathcal{V}_1(t, p) = & p\Phi(d_1(L)) - Ke^{-r(T-t)}\Phi(d_2(L)) \\ & + \alpha_D \eta_D(t) p^{-1/\beta_D} \Phi(-d_2(L) + \sigma\sqrt{T-t}/\beta_D) \\ & + \alpha_U \eta_U(t) p^{-1/\beta_U} \Phi(d_2(L) - \sigma\sqrt{T-t}/\beta_U) \end{aligned}$$

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Numerics: The Jump Uncertainty Price

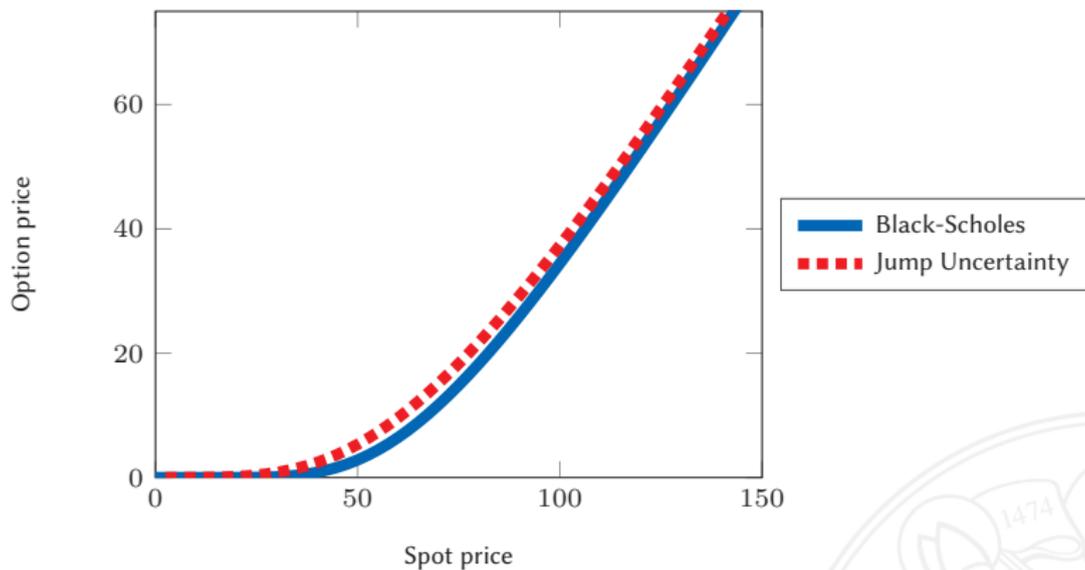


Figure: Jump uncertainty price for two-sided jumps; $\beta_D = -0.25$, $\beta_U = 0.25$.

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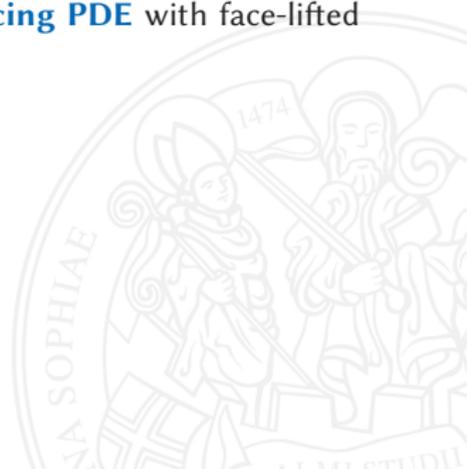
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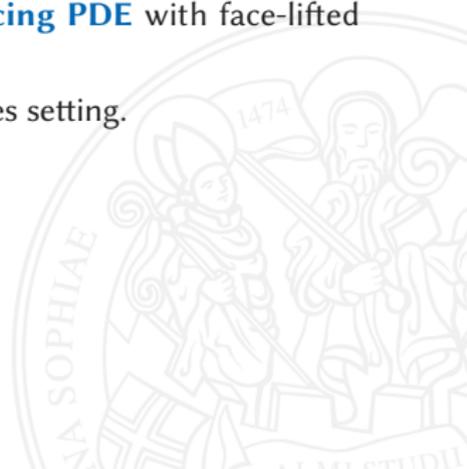
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Thank you very much for your attention!

Preprints on SSRN:

[1] Belak, C. and Menkens, O.: Pricing Contingent Claims under Jump Uncertainty, 2016.

[2] Menkens, O.: Pricing and Hedging of European Plain Vanilla Options under Jump Uncertainty, 2016.