Universität Trier

Backward Nonlinear Expectation Equations

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Joint work with **Thomas Seiferling** and **Frank Seifried**

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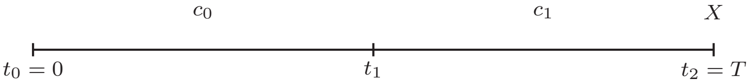
(5) These general results are then applied to study the **continuous-time limit** of recursive utility under uncertainty.

Recursive Utility and its Continuous-Time Limit

Recursive Utility (Kreps/Porteus 1978)



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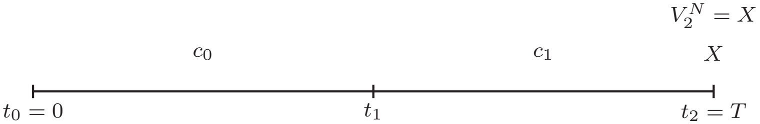

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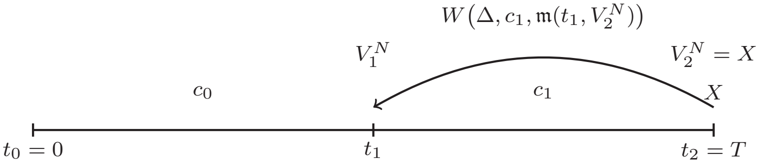

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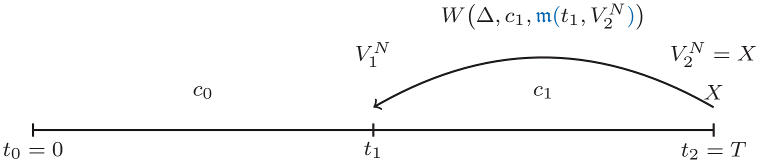

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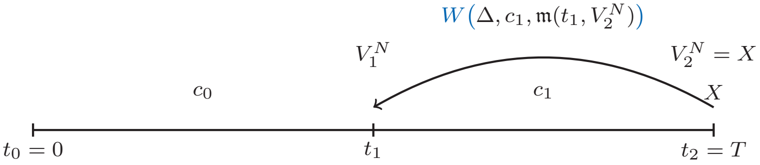

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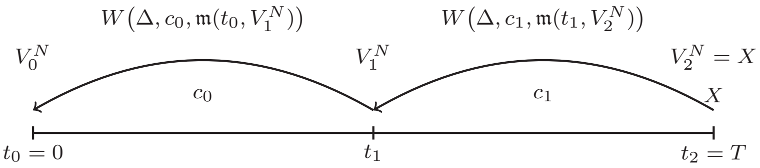

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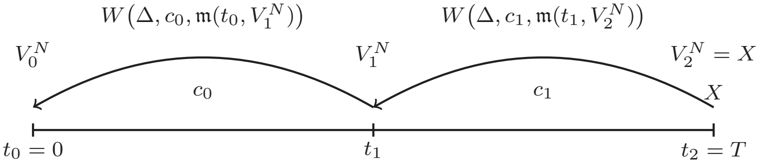

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$$\overset{\Delta=1}{\Rightarrow} \mathfrak{v}(c, X) = V_0^N = \ldots = U^{-1}\left(\mathbb{E}\left[\sum_{n=0}^{N-1} e^{-\delta t_n} U(c_n) + e^{-\delta T} U(X)\right]\right).$$

Stochastic Differential Utility (Duffie/Epstein 1992)

In a continuous-time setting, **stochastic differential utility** is defined axiomatically through

$$\mathfrak{v}(c, X) = V_0,$$

where $V = (V_t)_{t \in [0,T]}$ is given as the solution of the BSDE

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Example: If $f(c, v) = U(c) - \delta v$, then

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Relation between Recursive and Stochastic Differential Utility

Kraft/Seifried (2014): Suppose that

$$W(\Delta, c, v) \approx v + \Delta f(c, v) \quad \text{and} \quad \mathfrak{m}(t, V) = \mathbb{E}_t[V].$$

Then, under some technical conditions on f and (c, X), we have

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Aim: Extend the result of Kraft/Seifried to more general certainty equivalents.

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is expected to be a **backward nonlinear expectation equation** (BNEE) of the form

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Examples: g-Expectations, (random) G-Expectations, dynamic risk measures, ...

Appropriate Family of L^p spaces

Let Ω be a non-empty set. We take as given a family

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Remark: g-Expectations, G-Expectations and **random G-Expectations** are regular nonlinear expectations!

Existence and Uniqueness of BNEEs

Let $\xi \in L^p_T$ and consider the **general BNEE**

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Existence and Uniqueness of BNEEs

If $\mathcal{E}_t$ is **regular**, then the BNEE admits a unique solution in D^p.

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as well as the **discrete-time backward aggregation** $X_N^N \triangleq \xi$,

$$X_k^N \triangleq \mathcal{E}_{t_k}\left[g(t_k, \mathcal{E}_{t_k}[X_{k+1}^N])\Delta_{k+1}^N + X_{k+1}^N\right], \qquad k = N-1, \ldots, 0,$$

for a partition $\Delta^N : 0 = t_1 < \ldots < t_N = T$, $\Delta_k^N \triangleq t_k - t_{k-1}$. Assume that $(\Delta^N)_{N\in\mathbb{N}}$ is **refining** and **infinitesimal**.

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for a partition $\Delta^N : 0 = t_1 < \ldots < t_N = T$, $\Delta_k^N \triangleq t_k - t_{k-1}$. Assume that $(\Delta^N)_{N \in \mathbb{N}}$ is **refining** and **infinitesimal**.

Discrete-Time Approximation of BNEEs

If the mapping $t \mapsto g(t, Y)$ is càdlàg for every $Y \in L_T^p$, then

$$\max_{k=0,\ldots,N} \|X_k^N - X_{t_k}\|_{L,p} \to 0 \qquad \text{as } N \to \infty.$$

Application to Recursive Utility

Let (c, X) be a consumption plan. We define the **stochastic differential utility** process V under nonlinear expectations as

$$V_t = \mathcal{E}_t \left[\int_t^T f(c_s, V_s)\, \mathrm{d}s + X \right], \qquad t \in [0, T]$$

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$$W(\Delta, c, v) = v + \Delta f(c, u) + \epsilon(\Delta),$$

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Convergence of Recursive Utility

If $\mathcal{E}_t$ is regular, then under some technical assumptions on f and ϵ we have

$$\max_{k=0,\dots,N} \|V_k^N - V_{t_k^N}\|_{L,p} \to 0, \qquad \text{as } N \to \infty.$$

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Thank you very much for your attention!