Worst-Case Portfolio Optimization

Transaction Costs and Bubbles

Christoph Belak

Department of Mathematics
University of Kaiserslautern

March 17, 2015

Overview

(1) The Basics of Optimal Investment

Overview

(1) The Basics of Optimal Investment

(2) The Worst-Case Approach to Crash Modeling

Overview

(1) The Basics of Optimal Investment

(2) The Worst-Case Approach to Crash Modeling

(3) Portfolio Optimization with Transaction Costs

(1) The Basics of Optimal Investment

(2) The Worst-Case Approach to Crash Modeling

(3) Portfolio Optimization with Transaction Costs

The Fundamental Question

How do we invest our money optimally in a financial market?

How do we invest our money optimally in a financial market?

We need a model for the financial market:

We need a model for the financial market:

- Assets, asset prices and wealth dynamics

We need a model for the financial market:

- Assets, asset prices and wealth dynamics
- Trading strategies and trading constraints

We need a model for the financial market:

- Assets, asset prices and wealth dynamics
- Trading strategies and trading constraints
- Market frictions (transaction costs, taxes, liquidity, ...)

The Building Blocks of our Models

We need a model for the financial market:

- Assets, asset prices and wealth dynamics
- Trading strategies and trading constraints
- Market frictions (transaction costs, taxes, liquidity, ...)

We also need a model for the investor's preferences:

The Building Blocks of our Models

We need a model for the financial market:

- Assets, asset prices and wealth dynamics
- Trading strategies and trading constraints
- Market frictions (transaction costs, taxes, liquidity, ...)

We also need a model for the investor's preferences:

- Investment objective

The Building Blocks of our Models

We need a model for the financial market:

- Assets, asset prices and wealth dynamics
- Trading strategies and trading constraints
- Market frictions (transaction costs, taxes, liquidity, ...)

We also need a model for the investor's preferences:

- Investment objective
- Risk preferences

The Building Blocks of our Models

We need a model for the financial market:

- Assets, asset prices and wealth dynamics
- Trading strategies and trading constraints
- Market frictions (transaction costs, taxes, liquidity, ...)

We also need a model for the investor's preferences:

- Investment objective
- Risk preferences

In this talk, we are interested in an investor

We need a model for the financial market:

- Assets, asset prices and wealth dynamics
- Trading strategies and trading constraints
- Market frictions (transaction costs, taxes, liquidity, ...)

We also need a model for the investor's preferences:

- Investment objective
- Risk preferences

In this talk, we are interested in an investor

- ... who wants to make a long-term investment,

We need a model for the financial market:

- Assets, asset prices and wealth dynamics
- Trading strategies and trading constraints
- Market frictions (transaction costs, taxes, liquidity, ...)

We also need a model for the investor's preferences:

- Investment objective
- Risk preferences

In this talk, we are interested in an investor

- ... who wants to make a long-term investment,
- ... aims at maximizing her wealth,

The Building Blocks of our Models

We need a model for the financial market:

- Assets, asset prices and wealth dynamics
- Trading strategies and trading constraints
- Market frictions (transaction costs, taxes, liquidity, ...)

We also need a model for the investor's preferences:

- Investment objective
- Risk preferences

In this talk, we are interested in an investor

- ... who wants to make a long-term investment,
- ... aims at maximizing her wealth,
- ... is risk-averse,

We need a model for the financial market:

- Assets, asset prices and wealth dynamics
- Trading strategies and trading constraints
- Market frictions (transaction costs, taxes, liquidity, ...)

We also need a model for the investor's preferences:

- Investment objective
- Risk preferences

In this talk, we are interested in an investor

- ... who wants to make a long-term investment,
- ... aims at maximizing her wealth,
- ... is risk-averse,
- ... acknowledges the possibility of severe market crashes,

We need a model for the financial market:

- Assets, asset prices and wealth dynamics
- Trading strategies and trading constraints
- Market frictions (transaction costs, taxes, liquidity, ...)

We also need a model for the investor's preferences:

- Investment objective
- Risk preferences

In this talk, we are interested in an investor

- ... who wants to make a long-term investment,
- ... aims at maximizing her wealth,
- ... is risk-averse,
- ... acknowledges the possibility of severe market crashes,
- ... faces transaction costs.

Example: The Merton Model (1969)

The financial market: One bond/bank account, one stock.

Example: The Merton Model (1969)

The financial market: One bond/bank account, one stock.

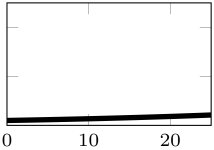
$$X_0 = x,$$

Here: x initial wealth, X_t wealth at time t

The financial market: One bond/bank account, one stock.

$$X_0 = x, \qquad \mathrm{d}\, X_t = r(1 - \pi_t) X_t \,\mathrm{d}\, t$$

Here: x initial wealth, X_t wealth at time t, π fraction of wealth invested in the stock, r interest rate

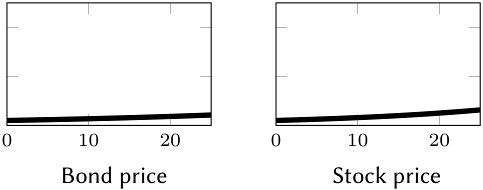


Bond price

The financial market: One bond/bank account, one stock.

$$X_0 = x, \qquad \mathrm{d}\,X_t = r(1-\pi_t)X_t\,\mathrm{d}\,t + \alpha\pi_t X_t\,\mathrm{d}\,t$$

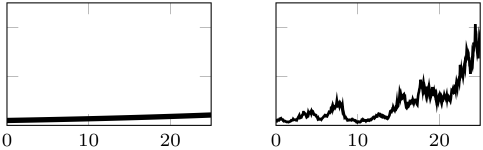
Here: x initial wealth, X_t wealth at time t, π fraction of wealth invested in the stock, r interest rate, $\alpha > r$ drift



The financial market: One bond/bank account, one stock.

$$X_0 = x, \qquad \mathrm{d}\, X_t = r(1-\pi_t)X_t\,\mathrm{d}\, t + \alpha\pi_t X_t\,\mathrm{d}\, t + \sigma\pi_t X_t\,\mathrm{d}\, W_t.$$

Here: x initial wealth, X_t wealth at time t, π fraction of wealth invested in the stock, r interest rate, $\alpha > r$ drift and σ volatility of the stock.



Bond price   Stock price

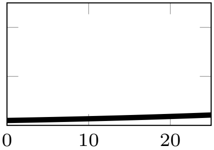
The financial market: One bond/bank account, one stock.

$$X_0 = x, \qquad \mathrm{d}\,X_t = r(1-\pi_t)X_t\,\mathrm{d}\,t + \alpha\pi_t X_t\,\mathrm{d}\,t + \sigma\pi_t X_t\,\mathrm{d}\,W_t.$$

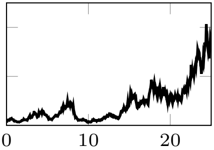
Here: x initial wealth, X_t wealth at time t, π fraction of wealth invested in the stock, r interest rate, $\alpha > r$ drift and σ volatility of the stock.

The investor's preferences:

$$\sup_{\pi} \mathbb{E}\Big[U_p\big(X_T^{\pi}\big)\Big],$$



Bond price



Stock price

The financial market: One bond/bank account, one stock.

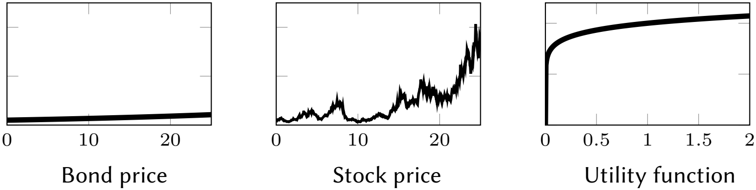
$$X_0 = x, \qquad \mathrm{d}\, X_t = r(1-\pi_t)X_t\,\mathrm{d}\,t + \alpha\pi_t X_t\,\mathrm{d}\,t + \sigma\pi_t X_t\,\mathrm{d}\,W_t.$$

Here: x initial wealth, X_t wealth at time t, π fraction of wealth invested in the stock, r interest rate, $\alpha > r$ drift and σ volatility of the stock.

The investor's preferences:

$$\sup_{\pi} \mathbb{E}\Big[U_p\big(X_T^{\pi}\big)\Big],$$

where $U_p(x) = x^p/p$, $p < 1$, denotes the investor's utility function.



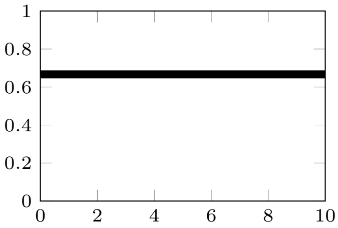
Solution of the Merton Model

The optimal strategy π_M in the Merton model is known explicitly and given by

$$\pi_M \triangleq \frac{\alpha - r}{(1-p)\sigma^2}.$$

The optimal strategy π_M in the Merton model is known explicitly and given by

$$\pi_M \triangleq \frac{\alpha - r}{(1-p)\sigma^2}.$$



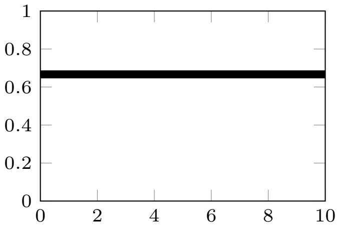
The Merton fraction π_M over time.

The optimal strategy π_M in the Merton model is known explicitly and given by

$$\pi_M \triangleq \frac{\alpha - r}{(1-p)\sigma^2}.$$

Model assumptions:

☑ Long-term investment



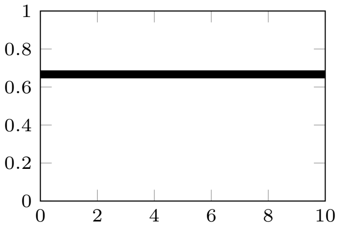
The Merton fraction π_M over time.

The optimal strategy π_M in the Merton model is known explicitly and given by

$$\pi_M \triangleq \frac{\alpha - r}{(1-p)\sigma^2}.$$

Model assumptions:

- ☑ Long-term investment
- ☑ Wealth maximization



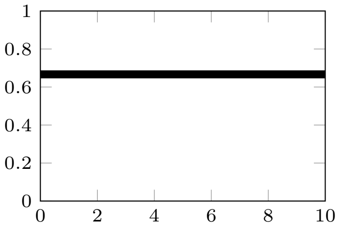
The Merton fraction π_M over time.

The optimal strategy π_M in the Merton model is known explicitly and given by

$$\pi_M \triangleq \frac{\alpha - r}{(1-p)\sigma^2}.$$

Model assumptions:

- ☑ Long-term investment
- ☑ Wealth maximization
- ☑ Risk aversion



The Merton fraction π_M over time.

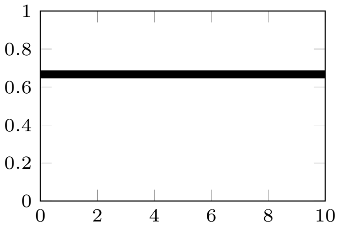
Solution of the Merton Model

The optimal strategy π_M in the Merton model is known explicitly and given by

$$\pi_M \triangleq \frac{\alpha - r}{(1-p)\sigma^2}.$$

Model assumptions:

- ☑ Long-term investment
- ☑ Wealth maximization
- ☑ Risk aversion
- ☐ Market crashes



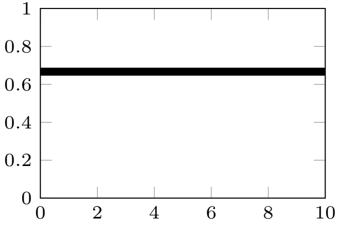
The Merton fraction π_M over time.

The optimal strategy π_M in the Merton model is known explicitly and given by

$$\pi_M \triangleq \frac{\alpha - r}{(1-p)\sigma^2}.$$

Model assumptions:

- ☑ Long-term investment
- ☑ Wealth maximization
- ☑ Risk aversion
- ☐ Market crashes
- ☐ Transaction costs



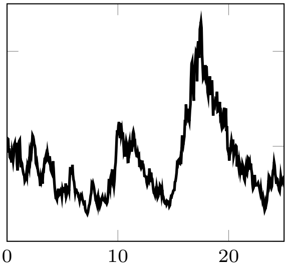
The Merton fraction π_M over time.

Idea: Let's add a jump component to the stock price dynamics!

$$\mathrm{d}\, X_t = r(1-\pi_t)X_t \,\mathrm{d}\, t + \alpha \pi_t X_t \,\mathrm{d}\, t + \sigma \pi_t X_t \,\mathrm{d}\, W_t$$

Idea: Let's add a jump component to the stock price dynamics!

$$\mathrm{d}\,X_t = r(1-\pi_t)X_t\,\mathrm{d}\,t + \alpha\pi_t X_t\,\mathrm{d}\,t + \sigma\pi_t X_t\,\mathrm{d}\,W_t$$

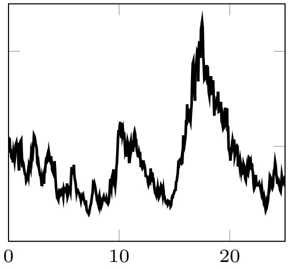


Stock price

Idea: Let's add a jump component to the stock price dynamics!

$$\mathrm{d}\,X_t = r(1-\pi_t)X_t\,\mathrm{d}\,t + \alpha\pi_t X_t\,\mathrm{d}\,t + \sigma\pi_t X_t\,\mathrm{d}\,W_t\ -\beta\pi_t X_t\,\mathrm{d}\,N_t,$$

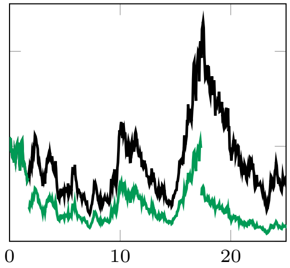
where N is a compensated Poisson process.



Idea: Let's add a jump component to the stock price dynamics!

$$\mathrm{d}\,X_t = r(1-\pi_t)X_t\,\mathrm{d}\,t + \alpha\pi_t X_t\,\mathrm{d}\,t + \sigma\pi_t X_t\,\mathrm{d}\,W_t\; -\beta\pi_t X_t\,\mathrm{d}\,N_t,$$

where N is a compensated Poisson process.



Stock price

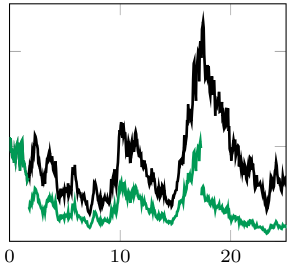
Idea: Let's add a jump component to the stock price dynamics!

$$\mathrm{d}\,X_t = r(1-\pi_t)X_t\,\mathrm{d}\,t + \alpha\pi_t X_t\,\mathrm{d}\,t + \sigma\pi_t X_t\,\mathrm{d}\,W_t \; -\beta\pi_t X_t\,\mathrm{d}\,N_t,$$

where N is a compensated Poisson process.

Drawbacks of this approach:

- The optimal strategy does not depend on the investment horizon.



Stock price

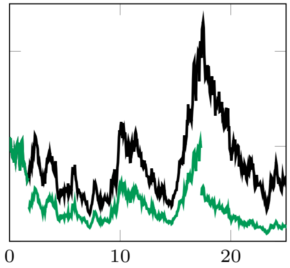
Idea: Let's add a jump component to the stock price dynamics!

$$\mathrm{d}\,X_t = r(1-\pi_t)X_t\,\mathrm{d}\,t + \alpha\pi_t X_t\,\mathrm{d}\,t + \sigma\pi_t X_t\,\mathrm{d}\,W_t \; -\beta\pi_t X_t\,\mathrm{d}\,N_t,$$

where N is a compensated Poisson process.

Drawbacks of this approach:

- The optimal strategy does not depend on the investment horizon.
- The losses are averaged across the investment period.



Stock price

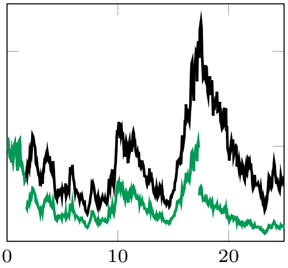
Idea: Let's add a jump component to the stock price dynamics!

$$\mathrm{d}\,X_t = r(1-\pi_t)X_t\,\mathrm{d}\,t + \alpha\pi_t X_t\,\mathrm{d}\,t + \sigma\pi_t X_t\,\mathrm{d}\,W_t \; - \beta\pi_t X_t\,\mathrm{d}\,N_t,$$

where N is a compensated Poisson process.

Drawbacks of this approach:

- The optimal strategy does not depend on the investment horizon.
- The losses are averaged across the investment period.
- It is hard to estimate the crash parameters.



Stock price

The Worst-Case Approach

Instead of assigning probabilities to crashes, only assume an upper bound on the number and the size of crashes.

Instead of assigning probabilities to crashes, only assume an upper bound on the number and the size of crashes.

We assume that the stock may crash by an unknown fraction $\beta \in [0, \beta^*]$ at an unknown stopping time $\tau \in [0, T] \cup \{\infty\}$.

Instead of assigning probabilities to crashes, only assume an upper bound on the number and the size of crashes.

We assume that the stock may crash by an unknown fraction $\beta \in [0, \beta^*]$ at an unknown stopping time $\tau \in [0, T] \cup \{\infty\}$. That is

$$X^\pi_\tau = X^\pi_{\tau-} - \beta \pi_\tau X^\pi_{\tau-} \qquad \text{on } \{\tau < \infty\}.$$

Instead of assigning probabilities to crashes, only assume an upper bound on the number and the size of crashes.

We assume that the stock may crash by an unknown fraction $\beta \in [0, \beta^*]$ at an unknown stopping time $\tau \in [0, T] \cup \{\infty\}$. That is

$$X^{\pi}_{\tau} = X^{\pi}_{\tau-} - \beta \pi_{\tau} X^{\pi}_{\tau-} \qquad \text{on } \{\tau < \infty\}.$$

We make no assumption about the distribution of (τ, β). Instead, we assume that the investor is extremely risk averse with respect to a crash:

$$\sup_{\pi} \inf_{\tau, \beta} \mathbb{E}\left[U_p\left(X^{\pi,\tau,\beta}_{T}\right)\right].$$

The stochastic optimization problem

$$\sup_{\pi} \inf_{\tau,\beta} \mathbb{E}\left[U_p\left(X_T^{\pi,\tau,\beta}\right)\right]$$

can be understood as a game between the investor and some opponent.

The stochastic optimization problem

$$\sup_{\pi} \inf_{\tau,\beta} \mathbb{E}\left[U_p\left(X_T^{\pi,\tau,\beta}\right)\right]$$

can be understood as a game between the investor and some opponent.

The investor faces a balancing problem between a good expected rate of return and crash exposure.

The stochastic optimization problem

$$\sup_{\pi} \inf_{\tau,\beta} \mathbb{E}\left[U_p\left(X_T^{\pi,\tau,\beta}\right)\right]$$

can be understood as a game between the investor and some opponent.

The investor faces a balancing problem between a good expected rate of return and crash exposure.

Literature: Korn/Wilmott (2002), Korn/Menkens (2005), Korn/Steffensen (2007), Seifried (2010), Desmettre/Korn/Seifried (2014), ...

Solution of the Worst-Case Problem

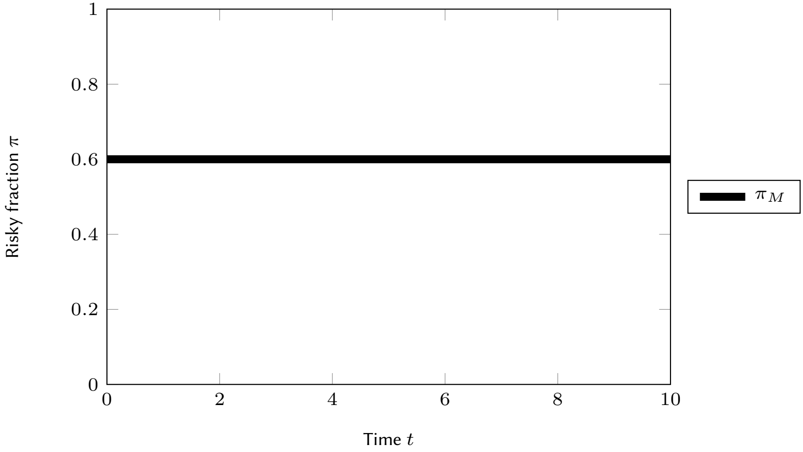


Figure: The solution of the worst-case problem.

Solution of the Worst-Case Problem

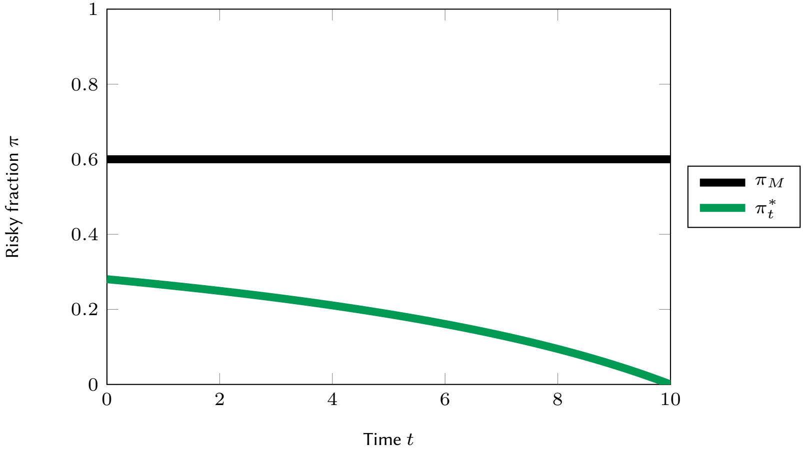


Figure: The solution of the worst-case problem.

The discussions up to now motivate the study of the following problems.

The discussions up to now motivate the study of the following problems.

Problem 1

Extend the worst-case model to allow for a random/unbounded number of crashes.

The discussions up to now motivate the study of the following problems.

Problem 1

Extend the worst-case model to allow for a random/unbounded number of crashes.

Problem 2

Study the worst-case model in the presence of transaction costs.

The discussions up to now motivate the study of the following problems.

Problem 1

Extend the worst-case model to allow for a random/unbounded number of crashes.

Problem 2

Study the worst-case model in the presence of transaction costs.

Overview

(1) The Basics of Optimal Investment

(2) The Worst-Case Approach to Crash Modeling

(3) Portfolio Optimization with Transaction Costs

(1) The Basics of Optimal Investment

(2) The Worst-Case Approach to Crash Modeling

(3) Portfolio Optimization with Transaction Costs

This section is based on:

BMS 15 Belak, Menkens, Sass (2015): *On the uniqueness of unbounded viscosity solutions arising in a terminal wealth problem with transaction costs.* Preprint.

BS 15 Belak, Sass (2015): *Finite-horizon optimal optimal investment with transaction costs: Construction of the optimal strategies.* Preprint.

BMS 14 Belak, Menkens, Sass (2014): *Worst-case portfolio optimization with proportional transaction costs.* To appear in Stochastics.

Our aim is to study the worst-case model in the presence of proportional transaction costs.

Our aim is to study the worst-case model in the presence of proportional transaction costs.

We denote by B the investor's wealth in the bond and let S denote the investor's wealth in the stock. In the presence of proportional transaction costs, the wealth evolves as

$$\begin{aligned} \mathrm{d}\,B_t &= rB_t\,\mathrm{d}\,t & B_0 &= b, \\ \mathrm{d}\,S_t &= \alpha S_t\,\mathrm{d}\,t + \sigma S_t\,\mathrm{d}\,W_t & S_0 &= s. \end{aligned}$$

We denote by B the investor's wealth in the bond and let S denote the investor's wealth in the stock. In the presence of proportional transaction costs, the wealth evolves as

$$\begin{aligned} \mathrm{d}\,B_t &= rB_t\,\mathrm{d}\,t - (1+\lambda)\,\mathrm{d}\,L_t & B_0 &= b, \\ \mathrm{d}\,S_t &= \alpha S_t\,\mathrm{d}\,t + \sigma S_t\,\mathrm{d}\,W_t + \mathrm{d}\,L_t & S_0 &= s. \end{aligned}$$

- L models the cumulative amount of money used for purchases of the stock.
- λ models the costs for purchases

We denote by B the investor's wealth in the bond and let S denote the investor's wealth in the stock. In the presence of proportional transaction costs, the wealth evolves as

$$
\begin{aligned}
\mathrm{d}\,B_t &= rB_t\,\mathrm{d}\,t - (1+\lambda)\,\mathrm{d}\,L_t + (1-\mu)\,\mathrm{d}\,M_t, && B_0 = b,\\
\mathrm{d}\,S_t &= \alpha S_t\,\mathrm{d}\,t + \sigma S_t\,\mathrm{d}\,W_t + \mathrm{d}\,L_t - \mathrm{d}\,M_t, && S_0 = s.
\end{aligned}
$$

- L models the cumulative amount of money used for purchases of the stock.
- M models the cumulative amount of money used for sales of the stock.
- λ and μ model the costs for purchases and sales, respectively.

We denote by B the investor's wealth in the bond and let S denote the investor's wealth in the stock. In the presence of proportional transaction costs, the wealth evolves as

$$\begin{aligned} \mathrm{d}\, B_t &= rB_t \,\mathrm{d}\, t - (1+\lambda)\,\mathrm{d}\, L_t + (1-\mu)\,\mathrm{d}\, M_t, && B_0 = b, \\ \mathrm{d}\, S_t &= \alpha S_t \,\mathrm{d}\, t + \sigma S_t \,\mathrm{d}\, W_t + \mathrm{d}\, L_t - \mathrm{d}\, M_t, && S_0 = s. \end{aligned}$$

- L models the cumulative amount of money used for purchases of the stock.
- M models the cumulative amount of money used for sales of the stock.
- λ and μ model the costs for purchases and sales, respectively.

The investor's wealth after liquidation of the stock position is

$$X_t = \begin{cases} B_t + (1-\mu)S_t, & \text{if } S_t > 0, \\ B_t + (1+\lambda)S_t, & \text{if } S_t \leq 0. \end{cases}$$

We denote by B the investor's wealth in the bond and let S denote the investor's wealth in the stock. In the presence of proportional transaction costs, the wealth evolves as

$$\begin{aligned} \mathrm{d}\,B_t &= rB_t\,\mathrm{d}\,t - (1+\lambda)\,\mathrm{d}\,L_t + (1-\mu)\,\mathrm{d}\,M_t, & B_0 &= b, \\ \mathrm{d}\,S_t &= \alpha S_t\,\mathrm{d}\,t + \sigma S_t\,\mathrm{d}\,W_t + \mathrm{d}\,L_t - \mathrm{d}\,M_t, & S_0 &= s. \end{aligned}$$

- L models the cumulative amount of money used for purchases of the stock.
- M models the cumulative amount of money used for sales of the stock.
- λ and μ model the costs for purchases and sales, respectively.

The investor's wealth after liquidation of the stock position is

$$X_t = \begin{cases} B_t + (1-\mu)S_t, & \text{if } S_t > 0, \\ B_t + (1+\lambda)S_t, & \text{if } S_t \leq 0. \end{cases}$$

We say that a trading strategy (L, M) is admissible if $X^{L,M} \geq 0$.

In the absence of crashes, the optimization problem is given by

$$\mathcal{V}_0(t,b,s) \triangleq \sup_{\varpi_0=(L,M)} \mathbb{E}_{t,b,s}\left[U_p\left(X_T^{\varpi_0}\right)\right].$$

In the absence of crashes, the optimization problem is given by

$$\mathcal{V}_0(t,b,s) \triangleq \sup_{\varpi_0=(L,M)} \mathbb{E}_{t,b,s}\Big[U_p\Big(X_T^{\varpi_0}\Big)\Big].$$

Literature: Magill/Constantinides (1976), Davis/Norman (1990), Shreve/Soner (1994), Akian/Séquier/Sulem (1995), Dai/Yi (2009), Herzog/Kunisch/Sass (2013), ...

In the absence of crashes, the optimization problem is given by

$$\mathcal{V}_0(t,b,s) \triangleq \sup_{\varpi_0=(L,M)} \mathbb{E}_{t,b,s}\Big[U_p\Big(X_T^{\varpi_0}\Big)\Big].$$

In the presence of at most one crash, the optimization problem is given by

$$\mathcal{V}_1(t,b,s) \triangleq \sup_{\varpi_1,\varpi_0} \inf_{\tau,\beta} \mathbb{E}_{t,b,s}\left[U_p\left(X_T^{\varpi_1,\varpi_0,\tau,\beta}\right)\right].$$

Literature: Magill/Constantinides (1976), Davis/Norman (1990), Shreve/Soner (1994), Akian/Séquier/Sulem (1995), Dai/Yi (2009), Herzog/Kunisch/Sass (2013), ...

Some Intuition on the Optimal Strategies

Before proceeding, let us think for a moment what we should expect from the optimal strategies.

Before proceeding, let us think for a moment what we should expect from the optimal strategies.

- The strategies π_M and π^* can no longer be optimal. They require infinite variation trading and would lead to immediate bankruptcy of the investor.

Before proceeding, let us think for a moment what we should expect from the optimal strategies.

- The strategies π_M and π^* can no longer be optimal. They require infinite variation trading and would lead to immediate bankruptcy of the investor.
- The costs penalize the volume of the transaction.

Before proceeding, let us think for a moment what we should expect from the optimal strategies.

- The strategies π_M and π^* can no longer be optimal. They require infinite variation trading and would lead to immediate bankruptcy of the investor.
- The costs penalize the volume of the transaction.
- We hence expect that the investor will try to make small trades for the transaction costs not to explode.

Some Intuition on the Optimal Strategies

Before proceeding, let us think for a moment what we should expect from the optimal strategies.

- The strategies π_M and π^* can no longer be optimal. They require infinite variation trading and would lead to immediate bankruptcy of the investor.
- The costs penalize the volume of the transaction.
- We hence expect that the investor will try to make small trades for the transaction costs not to explode.
- More precisely, we expect that the investor will try to keep her risky fraction in a neighborhood of π_M and π^*, respectively, and only make small trades to keep the risky fraction from moving too far away.

Before proceeding, let us think for a moment what we should expect from the optimal strategies.

- The strategies π_M and π^* can no longer be optimal. They require infinite variation trading and would lead to immediate bankruptcy of the investor.
- The costs penalize the volume of the transaction.
- We hence expect that the investor will try to make small trades for the transaction costs not to explode.
- More precisely, we expect that the investor will try to keep her risky fraction in a neighborhood of π_M and π^*, respectively, and only make small trades to keep the risky fraction from moving too far away.
- There should hence be three regions:
  A no-trade region, a buy region and a sell region.

The Dynamic Programming Principle

The key tool in studying the stochastic optimization problems we face is the so-called dynamic programming principle (DPP):

> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*
>
> — *Richard Bellman, 1957*

The key tool in studying the stochastic optimization problems we face is the so-called dynamic programming principle (DPP):

> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*
>
> — *Richard Bellman, 1957*

Theorem: Dynamic Programming [BMS 15]

For every $[t, T]$-valued stopping time θ, we have

$$\mathcal{V}_0(t, b, s) = \sup_{\varpi} \mathbb{E}\left[\mathcal{V}_0\left(\theta, B_\theta^{\varpi}, S_\theta^{\varpi}\right)\right]$$

The key tool in studying the stochastic optimization problems we face is the so-called dynamic programming principle (DPP):

> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*
>
> — *Richard Bellman, 1957*

Theorem: Dynamic Programming [BMS 15]

For every $[t, T]$-valued stopping time θ, we have

$$\mathcal{V}_0(t, b, s) = \sup_{\varpi} \mathbb{E}\left[\mathcal{V}_0\left(\theta, B_\theta^\varpi, S_\theta^\varpi\right)\right]$$

and

$$\mathcal{V}_1(t, b, s) = \sup_{\varpi_1} \inf_{\tau, \beta} \mathbb{E}\Big[\mathcal{V}_1\left(\theta, B_\theta^{\varpi_1}, S_\theta^{\varpi_1}\right) \mathbb{1}_{\{\theta < \tau\}} + \mathcal{V}_0\left(\tau, B_{\tau-}^{\varpi_1}, (1-\beta) S_{\tau-}^{\varpi_1}\right) \mathbb{1}_{\{\theta \geq \tau\}}\Big].$$

Arguing heuristically, the DPP can be used to derive partial differential equations for $\mathcal{V}_0$ and $\mathcal{V}_1$. These PDEs are the so-called dynamic programming equations.

Arguing heuristically, the DPP can be used to derive partial differential equations for $\mathcal{V}_0$ and $\mathcal{V}_1$. These PDEs are the so-called dynamic programming equations.

In the absence of crashes, we obtain the PDE

$$\min\left\{\mathcal{L}^{nt}\mathcal{V}_0(t,b,s),\mathcal{L}^{buy}\mathcal{V}_0(t,b,s),\mathcal{L}^{sell}\mathcal{V}_0(t,b,s)\right\}=0.$$

Arguing heuristically, the DPP can be used to derive partial differential equations for $\mathcal{V}_0$ and $\mathcal{V}_1$. These PDEs are the so-called dynamic programming equations.

In the absence of crashes, we obtain the PDE

$$\min\left\{\mathcal{L}^{nt}\mathcal{V}_0(t,b,s), \mathcal{L}^{buy}\mathcal{V}_0(t,b,s), \mathcal{L}^{sell}\mathcal{V}_0(t,b,s)\right\} = 0.$$

In the presence of at most once crash, the PDE is

$$\max\Big\{\mathcal{V}_1(t,b,s) - \mathcal{V}_0(t,b,(1-\beta^*)s), \\ \min\left\{\mathcal{L}^{nt}\mathcal{V}_1(t,b,s), \mathcal{L}^{buy}\mathcal{V}_1(t,b,s), \mathcal{L}^{sell}\mathcal{V}_1(t,b,s)\right\}\Big\} = 0.$$

Arguing heuristically, the DPP can be used to derive partial differential equations for $\mathcal{V}_0$ and $\mathcal{V}_1$. These PDEs are the so-called dynamic programming equations.

In the absence of crashes, we obtain the PDE

$$\min\left\{\mathcal{L}^{nt}\mathcal{V}_0(t,b,s), \mathcal{L}^{buy}\mathcal{V}_0(t,b,s), \mathcal{L}^{sell}\mathcal{V}_0(t,b,s)\right\} = 0.$$

In the presence of at most once crash, the PDE is

$$\max\Big\{\mathcal{V}_1(t,b,s) - \mathcal{V}_0(t,b,(1-\beta^*)s), \\ \min\left\{\mathcal{L}^{nt}\mathcal{V}_1(t,b,s), \mathcal{L}^{buy}\mathcal{V}_1(t,b,s), \mathcal{L}^{sell}\mathcal{V}_1(t,b,s)\right\}\Big\} = 0.$$

Formally, these PDEs are second-order, parabolic free boundary problems.

Classically, one proceeds as follows:

Classically, one proceeds as follows:

(1) Solve the PDEs and construct candidate optimal strategies along the way.

Classically, one proceeds as follows:

(1) Solve the PDEs and construct candidate optimal strategies along the way.

(2) Prove that the solutions of the PDEs coincide with $\mathcal{V}_0$ and $\mathcal{V}_1$, respectively, and that the candidate optimal strategies are indeed optimal.

Classically, one proceeds as follows:

(1) Solve the PDEs and construct candidate optimal strategies along the way.

(2) Prove that the solutions of the PDEs coincide with $\mathcal{V}_0$ and $\mathcal{V}_1$, respectively, and that the candidate optimal strategies are indeed optimal.

In our setting, we cannot expect that the PDEs admit classical solutions. But we can prove directly, that $\mathcal{V}_0$ and $\mathcal{V}_1$ solve the PDEs in a weak sense.

Classically, one proceeds as follows:

(1) Solve the PDEs and construct candidate optimal strategies along the way.

(2) Prove that the solutions of the PDEs coincide with $\mathcal{V}_0$ and $\mathcal{V}_1$, respectively, and that the candidate optimal strategies are indeed optimal.

In our setting, we cannot expect that the PDEs admit classical solutions. But we can prove directly, that $\mathcal{V}_0$ and $\mathcal{V}_1$ solve the PDEs in a weak sense.

Theorem: Viscosity Property of $\mathcal{V}_0$ and $\mathcal{V}_1$ [BMS 14,BMS 15]

$\mathcal{V}_0$ and $\mathcal{V}_1$ are the unique viscosity solutions of their respective PDEs.

Classically, one proceeds as follows:

(1) Solve the PDEs and construct candidate optimal strategies along the way.

(2) Prove that the solutions of the PDEs coincide with $\mathcal{V}_0$ and $\mathcal{V}_1$, respectively, and that the candidate optimal strategies are indeed optimal.

In our setting, we cannot expect that the PDEs admit classical solutions. But we can prove directly, that $\mathcal{V}_0$ and $\mathcal{V}_1$ solve the PDEs in a weak sense.

Theorem: Viscosity Property of $\mathcal{V}_0$ and $\mathcal{V}_1$ [BMS 14,BMS 15]

$\mathcal{V}_0$ and $\mathcal{V}_1$ are the unique viscosity solutions of their respective PDEs.

Remark: This approach enables us to study the PDEs numerically, but it does not establish existence of the optimal strategies.

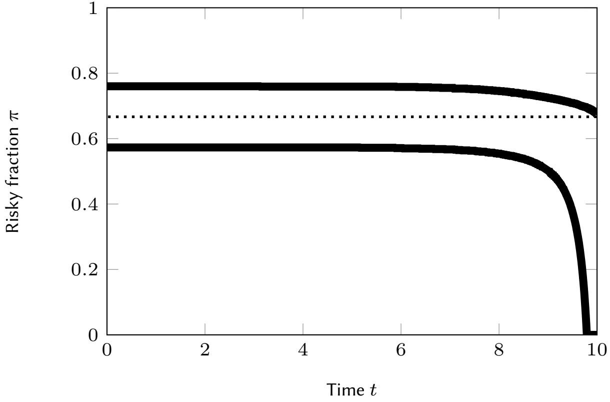


Figure: Optimal strategy with costs and without crashes.

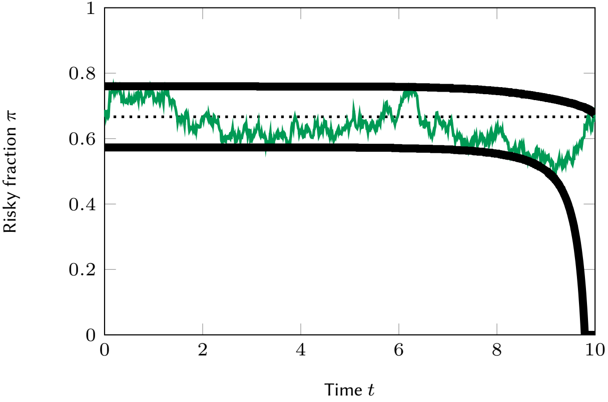


Figure: Optimal strategy with costs and without crashes.

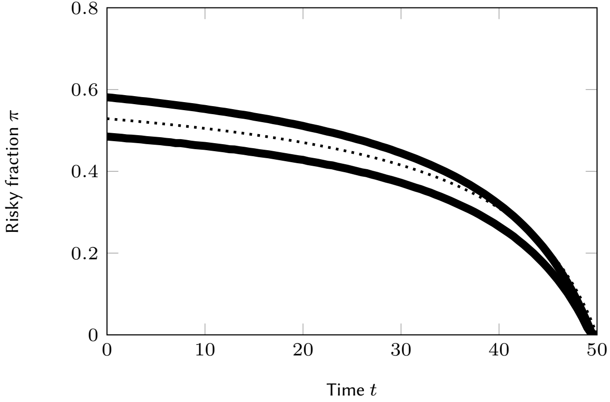


Figure: Optimal strategy with costs and with crashes.

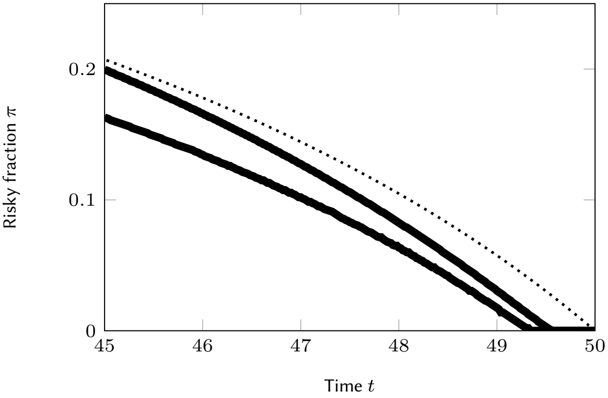


Figure: Optimal strategy with costs and with crashes.

Regarding the optimal strategy after the crash and the optimal strategy in the crash-free model, we can do more!

The Optimal Strategy in the Crash-Free Case

Regarding the optimal strategy after the crash and the optimal strategy in the crash-free model, we can do more!

Theorem: Construction of the Optimal Strategy [BS 15]

There exists a strategy ϖ^* which turns $(B^{\varpi^*}, S^{\varpi^*})$ into a diffusion reflected at the boundary of the no-trade region. Moreover, this strategy is optimal.

The Optimal Strategy in the Crash-Free Case

Regarding the optimal strategy after the crash and the optimal strategy in the crash-free model, we can do more!

Theorem: Construction of the Optimal Strategy [BS 15]

There exists a strategy ϖ^* which turns $(B^{\varpi^*}, S^{\varpi^*})$ into a diffusion reflected at the boundary of the no-trade region. Moreover, this strategy is optimal.

(1) Dai/Yi (2009) prove (partial) existence results of a classical solution of the PDE and establish properties of the free boundaries.

The Optimal Strategy in the Crash-Free Case

Regarding the optimal strategy after the crash and the optimal strategy in the crash-free model, we can do more!

Theorem: Construction of the Optimal Strategy [BS 15]

There exists a strategy ϖ^* which turns $(B^{\varpi^*}, S^{\varpi^*})$ into a diffusion reflected at the boundary of the no-trade region. Moreover, this strategy is optimal.

(1) Dai/Yi (2009) prove (partial) existence results of a classical solution of the PDE and establish properties of the free boundaries.

(2) The existence results turn out to be enough to construct a candidate optimal strategy ϖ^*.

The Optimal Strategy in the Crash-Free Case

Regarding the optimal strategy after the crash and the optimal strategy in the crash-free model, we can do more!

Theorem: Construction of the Optimal Strategy [BS 15]

There exists a strategy ϖ^* which turns $(B^{\varpi^*}, S^{\varpi^*})$ into a diffusion reflected at the boundary of the no-trade region. Moreover, this strategy is optimal.

(1) Dai/Yi (2009) prove (partial) existence results of a classical solution of the PDE and establish properties of the free boundaries.

(2) The existence results turn out to be enough to construct a candidate optimal strategy ϖ^*.

(3) Verify the optimality of ϖ^* by

Regarding the optimal strategy after the crash and the optimal strategy in the crash-free model, we can do more!

Theorem: Construction of the Optimal Strategy [BS 15]

There exists a strategy ϖ^* which turns $(B^{\varpi^*}, S^{\varpi^*})$ into a diffusion reflected at the boundary of the no-trade region. Moreover, this strategy is optimal.

(1) Dai/Yi (2009) prove (partial) existence results of a classical solution of the PDE and establish properties of the free boundaries.

(2) The existence results turn out to be enough to construct a candidate optimal strategy ϖ^*.

(3) Verify the optimality of ϖ^* by

  (a) ... writing down a suitable class $\mathbb{H}$ of superharmonic functions,

Regarding the optimal strategy after the crash and the optimal strategy in the crash-free model, we can do more!

Theorem: Construction of the Optimal Strategy [BS 15]

There exists a strategy ϖ^* which turns $(B^{\varpi^*}, S^{\varpi^*})$ into a diffusion reflected at the boundary of the no-trade region. Moreover, this strategy is optimal.

(1) Dai/Yi (2009) prove (partial) existence results of a classical solution of the PDE and establish properties of the free boundaries.

(2) The existence results turn out to be enough to construct a candidate optimal strategy ϖ^*.

(3) Verify the optimality of ϖ^* by

  (a) ... writing down a suitable class $\mathbb{H}$ of superharmonic functions,

  (b) ... proving that every member of $\mathbb{H}$ dominates $\mathcal{V}_0$,

Regarding the optimal strategy after the crash and the optimal strategy in the crash-free model, we can do more!

Theorem: Construction of the Optimal Strategy [BS 15]

There exists a strategy ϖ^* which turns $(B^{\varpi^*}, S^{\varpi^*})$ into a diffusion reflected at the boundary of the no-trade region. Moreover, this strategy is optimal.

(1) Dai/Yi (2009) prove (partial) existence results of a classical solution of the PDE and establish properties of the free boundaries.

(2) The existence results turn out to be enough to construct a candidate optimal strategy ϖ^*.

(3) Verify the optimality of ϖ^* by
   - (a) ... writing down a suitable class $\mathbb{H}$ of superharmonic functions,
   - (b) ... proving that every member of $\mathbb{H}$ dominates $\mathcal{V}_0$,
   - (c) ... defining $h^*(t,b,s) = \mathbb{E}_{t,b,s}[U_p(X_T^{\varpi^*})]$ and showing that $h^* \in \mathbb{H}$.

Conclusion:

Conclusion:

- Our aim was to analyze a model which includes transaction costs and the possibility of severe market crashes.

Conclusion:

- Our aim was to analyze a model which includes transaction costs and the possibility of severe market crashes.
- We proved several results: The dynamic programming principle, existence and uniqueness of viscosity solutions of the dynamic programming equations, construction of the candidate optimal strategy, verification of the optimality of the candidate optimal strategy, ...

Conclusion:

- Our aim was to analyze a model which includes transaction costs and the possibility of severe market crashes.
- We proved several results: The dynamic programming principle, existence and uniqueness of viscosity solutions of the dynamic programming equations, construction of the candidate optimal strategy, verification of the optimality of the candidate optimal strategy, ...

What's missing in this talk:

Conclusion:

- Our aim was to analyze a model which includes transaction costs and the possibility of severe market crashes.
- We proved several results: The dynamic programming principle, existence and uniqueness of viscosity solutions of the dynamic programming equations, construction of the candidate optimal strategy, verification of the optimality of the candidate optimal strategy, ...

What's missing in this talk:

- We extended the classical worst-case model to allow for a random/unbounded number of crashes and the possibility of financial bubbles.

Conclusion:

- Our aim was to analyze a model which includes transaction costs and the possibility of severe market crashes.
- We proved several results: The dynamic programming principle, existence and uniqueness of viscosity solutions of the dynamic programming equations, construction of the candidate optimal strategy, verification of the optimality of the candidate optimal strategy, ...

What's missing in this talk:

- We extended the classical worst-case model to allow for a random/unbounded number of crashes and the possibility of financial bubbles.
- We solved this model completely, derived the optimal strategies and provided extensive numerical examples.

Conclusion:

- Our aim was to analyze a model which includes transaction costs and the possibility of severe market crashes.
- We proved several results: The dynamic programming principle, existence and uniqueness of viscosity solutions of the dynamic programming equations, construction of the candidate optimal strategy, verification of the optimality of the candidate optimal strategy, ...

What's missing in this talk:

- We extended the classical worst-case model to allow for a random/unbounded number of crashes and the possibility of financial bubbles.
- We solved this model completely, derived the optimal strategies and provided extensive numerical examples.
- The results of this part of the thesis are included in:

BCM 14a Belak, Christensen, Menkens (2014): *Worst-Case Optimal Investment with a Random Number of Crashes*. Statitics and Probability Letters, 90, 140 – 148.

BCM 14b Belak, Christensen, Menkens (2014): *Worst-Case Portfolio Optimization in a Market with Bubbles*. Preprint.

Randomizing the Maximum Number of Crashes

We randomize the maximum number of crashes by assuming that the market switches between crash-free and crash-threatened states.

We randomize the maximum number of crashes by assuming that the market switches between crash-free and crash-threatened states.

The wealth dynamics are given as before

$$dX_t = r(1-\pi_t)X_t dt + \alpha\pi_t X_t dt + \sigma\pi_t X_t dW_t.$$

Randomizing the Maximum Number of Crashes

We randomize the maximum number of crashes by assuming that the market switches between crash-free and crash-threatened states.

The wealth dynamics are given as before

$$dX_t = r(1-\pi_t)X_t dt + \alpha\pi_t X_t dt + \sigma\pi_t X_t dW_t.$$

The switching between states is modeled through a continuous-time Markov chain Z with finite state space $E = \{0, 1, \ldots, d\}$.

Randomizing the Maximum Number of Crashes

We randomize the maximum number of crashes by assuming that the market switches between crash-free and crash-threatened states.

The wealth dynamics are now state-dependent

$$dX_t = r_i(1 - \pi_t^i)X_t dt + \alpha_i \pi_t^i X_t dt + \sigma_i \pi_t^i X_t dW_t.$$

The switching between states is modeled through a continuous-time Markov chain Z with finite state space $E = \{0, 1, \ldots, d\}$.

We allow for state-dependent market parameters and the investor can choose a different strategy π^i for each state.

Schematic of the Model

Z in state $\mathbf{0}$

$\hookrightarrow$ No crash possible

Schematic of the Model

Z in state $\mathbf{0}$

$\hookrightarrow$ No crash possible

$\downarrow$

Z jumps to state i

$\hookrightarrow$ Investor receives warning
$\hookrightarrow$ Crash of maximum size $\beta_i^* \geq 0$ possible

Schematic of the Model

$\boldsymbol{Z}$ in state $\mathbf{0}$

$\hookrightarrow$ No crash possible

$\downarrow$

$\boldsymbol{Z}$ jumps to state $\boldsymbol{i}$

$\hookrightarrow$ Investor receives warning
$\hookrightarrow$ Crash of maximum size $\beta_i^* \geq 0$ possible

$\swarrow$

Crash $(\boldsymbol{\tau}, \boldsymbol{\beta})$ occurs

$\hookrightarrow$ Stock price crashes by a fraction of $\beta \leq \beta_i^*$
$\hookrightarrow$ Z jumps back to state 0

Schematic of the Model

Z in state 0

↪ No crash possible

↓

Z jumps to state i

↪ Investor receives warning
↪ Crash of maximum size $\beta_i^* \geq 0$ possible

↙ ↘

Crash (τ, β) occurs

↪ Stock price crashes by a fraction of $\beta \leq \beta_i^*$
↪ Z jumps back to state 0

Z jumps to state j

↪ Investor receives new information
↪ Crash of maximum size $\beta_j^* \geq 0$ possible

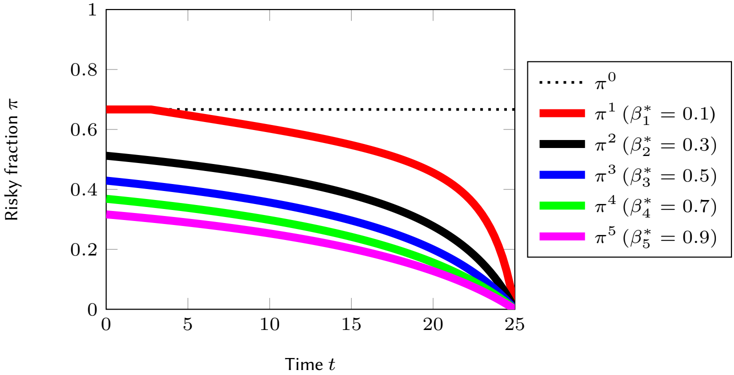


Figure: Optimal strategies in the case of a random number of crashes.

Optimal Strategies: Example II

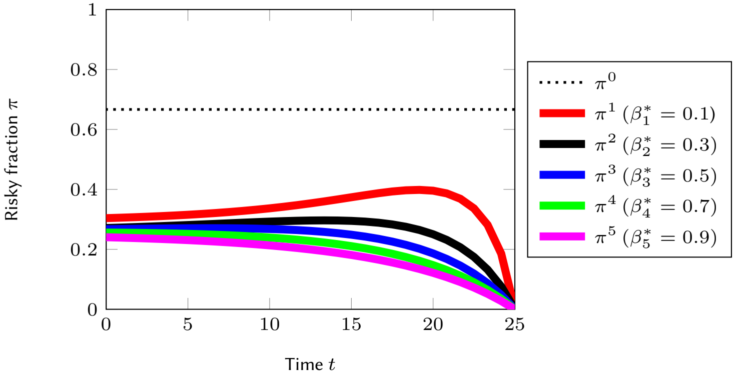


Figure: Optimal strategies in the case of a random number of crashes.

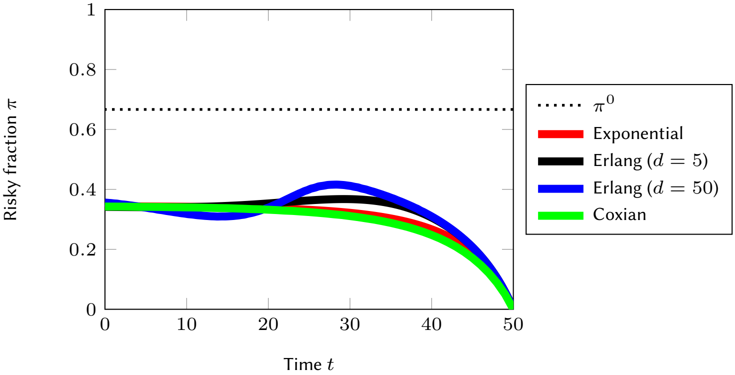


Figure: Optimal strategies in the case of a random number of crashes.

The Dynamic Programming Equation for $\beta_i^* = 0$

Denote by $(q_{i,j})_{0 \le i,j \le d}$ the generator matrix of Z_t and let

$$\mathcal{L}_i^\pi \mathcal{V} \triangleq \frac{\partial}{\partial t}\mathcal{V} + \alpha_i \pi x \frac{\partial}{\partial x}\mathcal{V} + \frac{1}{2}\sigma_i^2 \pi^2 x^2 \frac{\partial^2}{\partial x^2}\mathcal{V}, \quad i = 0, \ldots, d.$$

Dynamic Programming Equation for $\beta_i^* = 0$

The value function $\mathcal{V}(\cdot, \cdot, i)$ and the corresponding optimal strategy $\pi^{*,i}$ in state i can be determined by solving

$$0 = \sup_\pi \left[\mathcal{L}_i^\pi \mathcal{V}(t, x, i) + \sum_{j=0}^{d} q_{i,j} \mathcal{V}(t, x, j) \right].$$

Standard arguments show that the **optimal strategy** is

$$\pi_t^{i,*} = \pi_M^i \triangleq \frac{\alpha_i}{(1-p)\sigma_i^2}.$$

The Dynamic Programming Equation for $\beta_i^* > 0$

Define the following sets:

$$A_1 \triangleq \Big\{\pi \in \mathcal{K} : \mathcal{V}(t,x,i) \leq \mathcal{V}(t,(1-\beta_i^*\pi)x,0)\Big\},$$

$$A_2 \triangleq \Big\{\pi \in \mathcal{K} : \mathcal{L}_i^\pi \mathcal{V}(t,x,i) + \sum_{j=1}^{d} q_{i,j}\mathcal{V}(t,x,j) \geq 0\Big\}.$$

The value function $\mathcal{V}(\cdot,\cdot,i)$ and the corresponding optimal strategy $\pi^{*,i}$ in state i can be determined by solving

$$0 \leq \sup_{\pi \in A_1}\Big[\mathcal{L}_i^\pi \mathcal{V}(t,x,i) + \sum_{j=1}^{d} q_{i,j}\mathcal{V}(t,x,j)\Big],$$

$$0 \leq \sup_{\pi \in A_2}\Big[\mathcal{V}(t,(1-\beta_i^*\pi)x,0) - \mathcal{V}(t,x,i)\Big],$$

$$0 = \sup_{\pi \in A_1}\Big[\mathcal{L}_i^\pi \mathcal{V}(t,x,i) + \sum_{j=1}^{d} q_{i,j}\mathcal{V}(t,x,j)\Big] \cdot \sup_{\pi \in A_2}\Big[\mathcal{V}(t,(1-\beta_i^*\pi)x,0) - \mathcal{V}(t,x,i)\Big].$$

The Optimal Strategies in the Bubble Model

The optimal strategy in state i with $\beta_i^* > 0$ is given by $\pi_t^{i,*} \triangleq \min\{\pi_M^i, \pi_t^{i,\mathrm{ind}}\}$, where $\pi^{i,\mathrm{ind}}$ solves the differential equation

$$\frac{\partial}{\partial t}\pi_t^{i,\mathrm{ind}} = \frac{1}{\beta_i^*}(1-\pi_t^{i,\mathrm{ind}}\beta_i^*)\Bigg[\Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2\left(\pi_t^{i,\mathrm{ind}} - \pi_M^i\right)^2 + \frac{1}{p}\sum_{j=0}^{d} q_{i,j}\frac{f_j(t)}{f_0(t)}(1-\pi_t^{i,\mathrm{ind}}\beta_i^*)^{-p} - \frac{1}{p}\sum_{\substack{j=0\\ j\neq i}}^{d} q_{0,j}\frac{f_j(t)}{f_0(t)} - q_{0,i}\frac{1}{p}(1-\pi_t^{i,\mathrm{ind}}\beta_i^*)^{p}\Bigg].$$

Here, $\Psi_i \triangleq \frac{1}{2}\frac{\alpha_i^2}{(1-p)\sigma_i^2}$, and the functions f_i solve the system of ODEs

$$\frac{\partial}{\partial t}f_i(t) = -p\alpha_i \min\left\{\frac{1}{\beta_i^*}\left(1-\left[\frac{f_i(t)}{f_0(t)}\right]^{1/p}\right), \pi_M^i\right\} f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t) + \frac{1}{2}p(1-p)\sigma_i^2\left[\min\left\{\frac{1}{\beta_i^*}\left(1-\left[\frac{f_i(t)}{f_0(t)}\right]^{1/p}\right), \pi_M^i\right\}\right]^2 f_i(t).$$