Backward Nonlinear Expectation Equations

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Joint work with **Frank Seifried** and **Thomas Seiferling**

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BNEEs can be regarded as the equivalent of backward stochastic differential equations in a nonlinear expectation setting.

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Appropriate Family of L^p spaces

Let Ω be a non-empty set. We take as given a family

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In the sequel we write $L^p \triangleq L_T^p$ for simplicity.

Processes and corresponding Spaces

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(4) $(D^p, \|\cdot\|_{S,p})$ to be the Banach space of all $X \in S^p$ such that the mapping $[0, T] \to L^p$, $t \mapsto X_t$ is càdlàg.

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Existence and Uniqueness of BNEEs

Let $\xi \in L^p$ and consider the **general BNEE**

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Comparison Theorem

Let $\mathcal{E}_t$ be a **measurable** nonlinear expectation and assume that

$$X_t^1 \leq \mathcal{E}_t\left[\int_t^T g_1(s, X_s^1)\, ds + \xi_1\right], \quad t \in [0, T], \qquad \textbf{(BNEE-subsolution)}$$

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for a partition $\Delta^N : 0 = t_1 < \ldots < t_N = T$, $\Delta_k^N \triangleq t_k - t_{k-1}$. Assume that $(\Delta^N)_{N\in\mathbb{N}}$ is **refining** and **infinitesimal**.

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Convergence Theorem

If $t \mapsto g(t, Y)$ is càdlàg for every $Y \in L^p$, then

$$\max_{k=0,\ldots,N} \|X_k^N - X_{t_k}\|_{L,p} \to 0, \qquad \text{as } N \to \infty.$$

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