

Worst-Case Portfolio Optimization

Bubbles and Transaction Costs

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Based on joint work with **Sören Christensen**, **Olaf Menkens** and **Jörn Sass**.

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Overview

- 1 Optimal Investment in Continuous Time
- 2 The Worst-Case Approach to Market Crashes
- 3 Random Number of Crashes and Bubbles
- 4 Proportional Transaction Costs

Motivation

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- ... we want to make a long-term investment,
- ... we want to maximize our wealth at the end of the investment period,
- ... the market may possibly crash.

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The function U_p is called the **utility function** and models the investor's **risk preferences**. We assume that

$$U_p(x) = \begin{cases} \frac{1}{p} x^p, & \text{if } p < 1, p \neq 0, \\ \log(x), & \text{if } p = 0. \end{cases}$$

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- 2 We should expect the process $\mathcal{V}_0(u, X_{t,x}^{\pi^*}(u))$ to be an **honest martingale** for the optimal π^* .

The Hamilton-Jacobi-Bellman Equation

The dynamic programming principle leads to the following PDE – the **Hamilton-Jacobi-Bellman (HJB)** equation:

$$\frac{\partial}{\partial t} \mathcal{V}_0 + (r + (\alpha - r)\pi)x \frac{\partial}{\partial x} \mathcal{V}_0 + \frac{1}{2} \sigma^2 \pi^2 x^2 \frac{\partial^2}{\partial x^2} \mathcal{V}_0 \leq 0,$$
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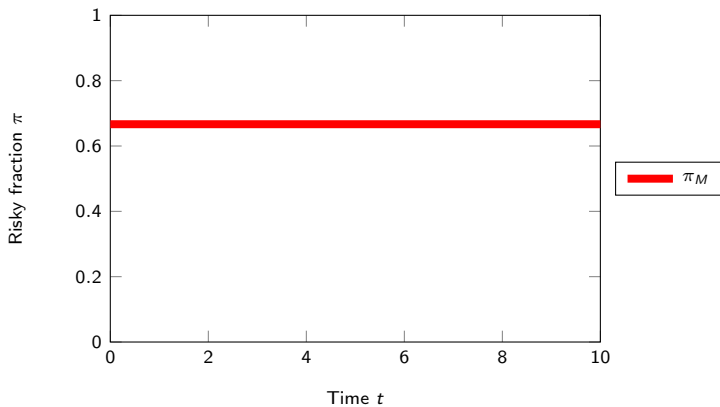
This equation is solved explicitly by

$$\mathcal{V}_0(t, x) = \frac{1}{p} x^p \exp \left\{ p \left(r - \frac{1}{2(1-p)} \frac{(\alpha - r)^2}{\sigma^2} \right) (T - t) \right\}$$

and the **optimal strategy** is given by

$$\pi_0^*(t) = \pi_M := \frac{\alpha - r}{(1 - p)\sigma^2}.$$

The Solution to the Merton Problem



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Observation: If a crash occurs close to the end of the investment period the investor will suffer substantial losses.

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The Worst-Case Approach of Korn/Wilmott (2002)

We assume that a crash is given by a pair $\vartheta = (\tau, \bar{\beta})$. At the stopping time τ , the **stock price crashes** by the fraction $\bar{\beta}$:

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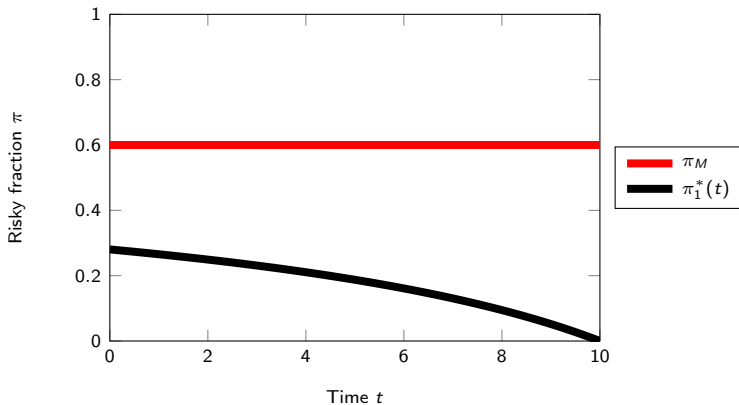
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Moreover, it is straightforward to verify that **this strategy is optimal!**

The Optimal Strategy in the Worst-Case Model



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- We allow for **state-dependent market parameters**.

Bubbles and Crashes

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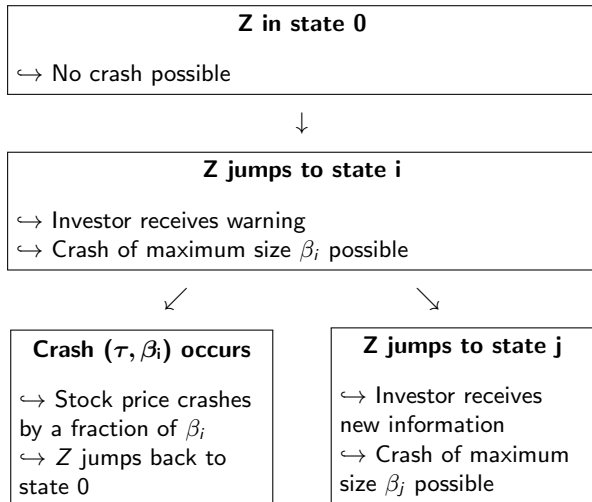


Crash (τ, β_i) occurs

↔ Stock price crashes
by a fraction of β_i

↔ Z jumps back to
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Whenever $\beta_i = 0$, the investor **does not have to fear a crash**. Following the solution approach as in the Merton model, we expect that $\mathcal{V}(t, x, i)$ solves an HJB equation of a similar form.

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$$\mathcal{L}_i^\pi \mathcal{V} = \frac{\partial}{\partial t} \mathcal{V} + (r_i + (\alpha_i - r_i)\pi)x \frac{\partial}{\partial x} \mathcal{V} + \frac{1}{2} \sigma_i^2 \pi^2 x^2 \frac{\partial^2}{\partial x^2} \mathcal{V}, \quad i = 0, \dots, d.$$

HJB Equation for States with $\beta_i = 0$

The **value function** $\mathcal{V}(\cdot, \cdot, i)$ and the corresponding **optimal strategy** π_i^* in state i with $\beta_i = 0$ can be determined by solving the following HJB equation:

$$0 \geq \mathcal{L}_i^\pi \mathcal{V}(t, x, i) + \sum_{j=0}^d q_{i,j} \mathcal{V}(t, x, j) .$$

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$$\mathcal{V}(t, x, i)$$

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Define the following sets:

$$A_1 := \left\{ \pi : \mathcal{V}(t, x, i) \leq \mathcal{V}(t, (1 - \pi\beta_i)x, 0) \right\},$$

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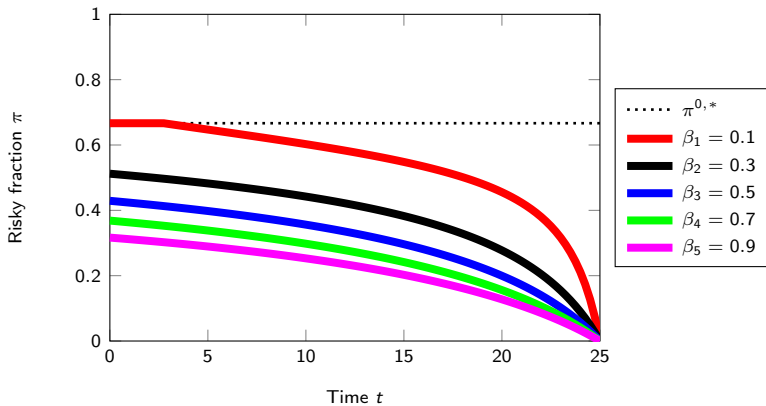
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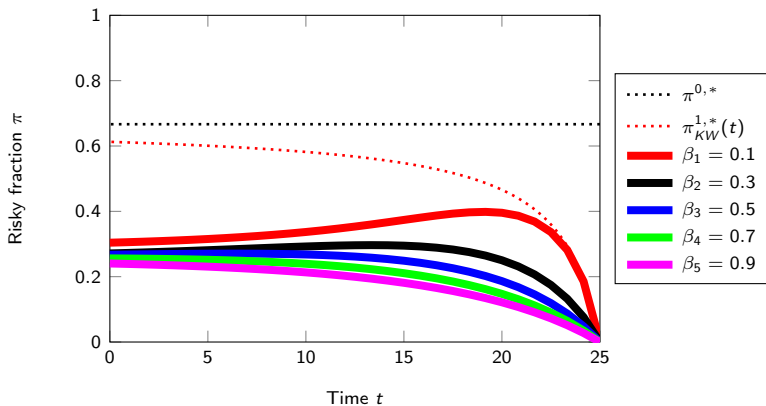
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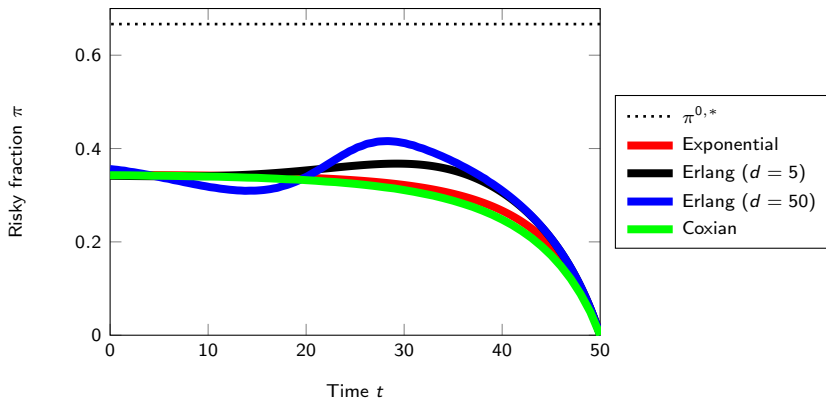
An Example



Another Example



One more...



Overview

- 1 Optimal Investment in Continuous Time
- 2 The Worst-Case Approach to Market Crashes
- 3 Random Number of Crashes and Bubbles
- 4 Proportional Transaction Costs**

Trading under Proportional Transaction Costs

Let us now consider the Korn/Wilmott model (one crash only) in the presence of **transaction costs**, i.e. we assume that the investor buys and sells shares of the stock at the price

$$(1 + \lambda)P_1(t) \quad \text{and} \quad (1 - \mu)P_1(t),$$

respectively, where $\lambda > 0$, $\mu \in (0, 1)$.

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Trading strategies are modeled as a pair (L, M) . The **wealth** invested in the bond and stock is assumed to be given by

$$\begin{aligned} dB(t) &= rB(t)dt & B(0) &= b, \\ dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t) & S(0) &= s. \end{aligned}$$

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The investor's **net wealth** X in this model is given by

$$X(t) = \begin{cases} B(t) + (1 - \mu)S(t), & \text{if } S(t) > 0, \\ B(t) + (1 + \lambda)S(t), & \text{if } S(t) \leq 0. \end{cases}$$

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$$\mathcal{V}_0(t, b, s) = \sup_{(L, M)} \mathbb{E} \left[U_p \left(X_{t, b, s}^{L, M}(T) \right) \right]$$

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$$\mathcal{V}_1(t, b, s) = \sup_{\varpi_1 = (L_1, M_1)} \inf_{\varpi_0 = (L_0, M_0)} \inf_{\tau \in \mathcal{B}(t)} \mathbb{E} \left[U_p \left(X_{t, b, s}^{\varpi_1, \varpi_0, \tau}(T) \right) \right]$$

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- More precisely, we expect that the investor will try to keep her risky fraction **in a neighborhood of the Merton fraction** and only makes small trades to keep the risky fraction from moving too far away.
- There should hence be three regions: A **no-trading region**, a **buying region** and a **selling region**.

The HJB Equation in the Absence of Crashes

The HJB equation in the **absence of crashes** takes the following form

$$0 = \min \left\{ \mathcal{L}^{nt} \mathcal{V}_0(t, b, s), \mathcal{L}^{buy} \mathcal{V}_0(t, b, s), \mathcal{L}^{sell} \mathcal{V}_0(t, b, s) \right\},$$

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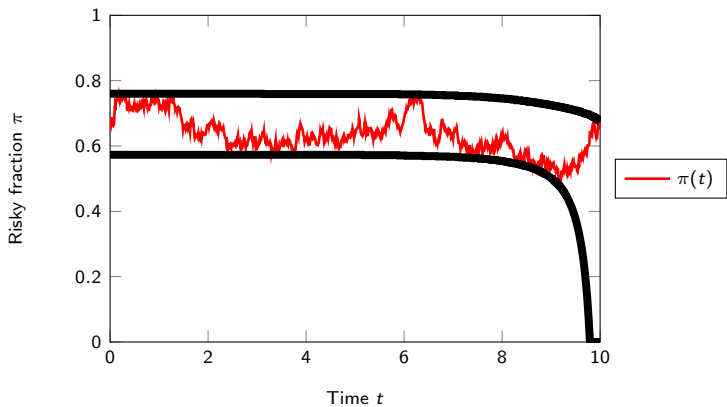
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But: We can show that the the value function is a weak (viscosity) solution of the HJB equation and determine the trading regions numerically.

A Numerical Example



The HJB Equation in the Presence of Crashes

In the **presence of crashes**, the HJB equation is given by

$$0 = \max \left\{ \mathcal{V}_1(t, b, s) - \mathcal{V}_0(t, b, (1 - \beta)s), \right. \\ \left. \min \left\{ \mathcal{L}^{nt} \mathcal{V}_1(t, b, s), \mathcal{L}^{buy} \mathcal{V}_1(t, b, s), \mathcal{L}^{sell} \mathcal{V}_1(t, b, s) \right\} \right\}.$$

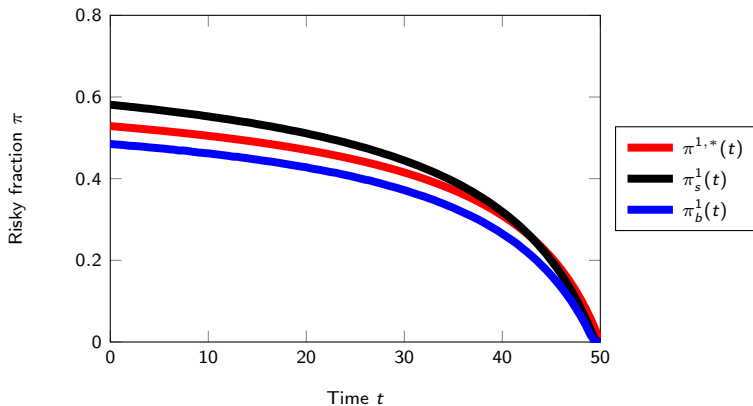
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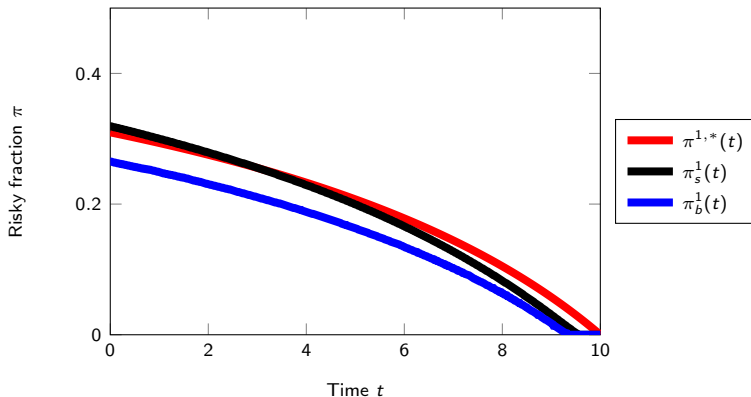
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As in the case without crashes, we can show that \mathcal{V}_1 solves this equation in the weak (viscosity) sense.

Trading Regions in the Presence of Crashes: Long term



Trading Regions in the Presence of Crashes: Short term



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- The assumption of a fixed number of crashes can be relaxed and we can even model **financial bubbles**.
- In the presence of **transaction costs**, the investor keeps her risky fraction in a region around the optimal no-cost strategy.
- In the presence of crashes, it may sometimes **not be optimal** to invest any money in the stock at all.

Literature

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