Valuation of Contingent Claims under Jump Uncertainty

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Literature: Hua and Wilmott (1996) and Mönnig (2012).

The market model

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- The investor can **observe the price jump** and adjust her trading strategy afterwards.
- We denote the **set of all such jump scenarios** by $\mathcal{B}$.

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In this setting, given a jump scenario $\iota = (\tau, \beta)$, the wealth evolves as

$$\begin{aligned} dX_t &= (rX_t + \theta\sigma\pi_t)\, dt + \sigma\pi_t\, dW_t, && t \in [0, \tau), \\ X_\tau &= X_{\tau-} - \beta\pi_{\tau-}, && \\ dX_t &= (rX_t + \theta\sigma\pi_t^\iota)\, dt + \sigma\pi_t^\iota\, dW_t, && t \in [\tau, T], \end{aligned}$$

where $\theta = (\alpha - r)\sigma$.

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Trading strategies are called **admissible**, if they lead to non-negative wealth in all jump scenarios $\iota \in \mathcal{B}$.

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Our aim is to price an **option** with **payoff** $\xi(S_T)$ (square-integrable, non-negative).

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Well-definedness of the worst-case price

Denote by $\check{V}(t,s)$ the arbitrage-free price of the option $\xi(S_T)$ in the corresponding **jump-free market** at time t with stock price $S_t = s$. The worst-case price of the option is **well-defined** if and only if there exists a pre-jump strategy π such that

$$X_t - \sup_{\beta \in [\underline{\beta}, \overline{\beta}]} \left[\beta \pi_t + \check{V}\big(t, (1-\beta) S_{t-}\big) \right] \geq 0, \qquad \text{for all } t, \mathbb{P}\text{-a.s.}$$

BSDE characterization of the worst-case price

We denote the time-t worst-case price of the option $\xi(S_T)$ with $S_t = s$ by $V(t, s)$.

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BSDE characterization of the worst-case price

Assume that the worst-case price $V(t,s)$ is **well-defined**. Assume further that the fair price $\check{V}(t,s)$ in the corresponding jump-free market is **jointly continuous** in (t,s). Then the worst-case price $V(t,s)$ of $\xi(S_T)$ **exists** and is given as the **minimum solution** X_t of

$$dX_u = (rX_u + \theta\sigma\pi_u)\,du + \sigma\pi_u\,dW_u, \qquad\qquad X_T \geq \xi(S_T),$$

under the **constraint**

$$X_u - \sup_{\beta\in[\underline{\beta},\overline{\beta}]} \left[\beta\pi_u + \check{V}\big(u,(1-\beta)S_{u-}\big)\right] \geq 0 \qquad \text{for all } u, \mathbb{P}\text{-a.s.}$$

Literature: Peng (1999).

PDE characterization of the worst-case price

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Assume that the payoff $\xi(s)$ is a Lipschitz-continuous function of the stock price. Then V is the **unique viscosity solution** of

$$0 = \min\left\{ -V_t - \frac{1}{2}\sigma^2 s^2 V_{ss} - rsV_s + rV, V - \sup_{\beta \in [\underline{\beta}, \overline{\beta}]} \left[\beta s V_s + \check{V}(t, (1-\beta)s) \right] \right\}.$$

Face-lifting the payoff

One may expect the terminal condition of the pricing PDE to be $V(T, s) = \xi(s)$. However, since the jump-constraint has to be satisfied for $t < T$, this would lead to a **discontinuity at terminal time**.

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The terminal condition

Define

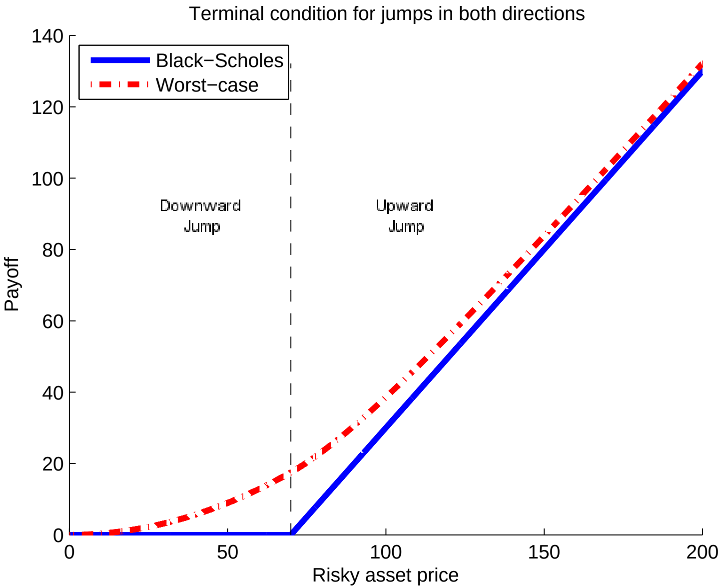
$$T^*(s) = \lim \sup_{t \to T, s' \to s} V(t,s'), \qquad T_*(s) = \lim \inf_{t \to T, s' \to s} V(t,s').$$

If $\xi(s)$ is Lipschitz-continuous, then $T(s) := T^*(s) = T_*(s)$ and T is the unique viscosity solution of

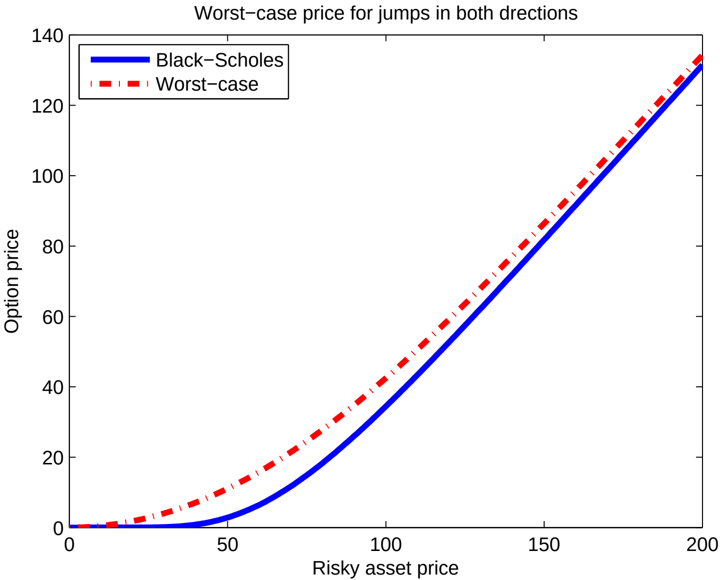
$$0 = \min\left\{ T(s) - \xi(s), T(s) - \sup_{\beta \in [\underline{\beta}, \overline{\beta}]} [\beta s T_s(s) + \xi((1-\beta)s)] \right\}.$$

Literature: Bouchard (2002).

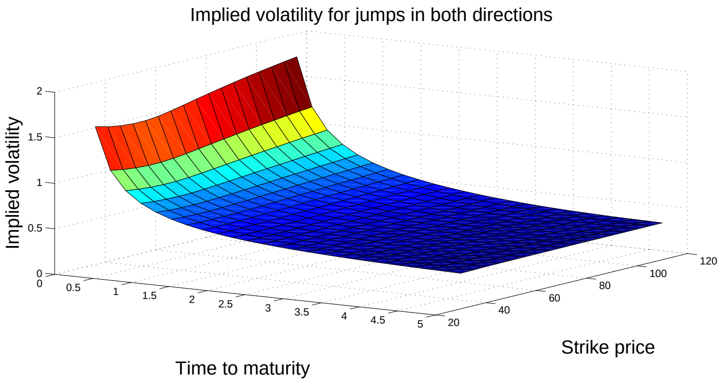
Numerical results: Terminal condition



Numerical results: Worst-case price



Numerical results: Implied volatility surface



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- Discuss risk management applications: What is the implied maximum jump size?

Thank you for your attention!!!