Optimal Terminal Wealth with Transaction Costs

and Uniqueness of Unbounded Viscosity Solutions

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Outline

The Market Model

We consider the following financial market. The investor can only invest in a risk-free asset ("**bond**") or a risky asset ("**stock**").

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Assume the wealth invested in the bond and stock, resp., follows

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The investor's **net wealth** X is then

$$X_t = \begin{cases} B_t + (1-\mu)S_t, & \text{if } S_t > 0, \\ B_t + (1+\lambda)S_t, & \text{if } S_t \leq 0. \end{cases}$$

Admissibility and Solvency

We say that a trading strategy (L, M) is **admissible**, if the corresponding net wealth process X is a.s. nonnegative.

This is equivalent to saying that the pair (B_t, S_t) stays inside the closure of the following **solvency cone**:

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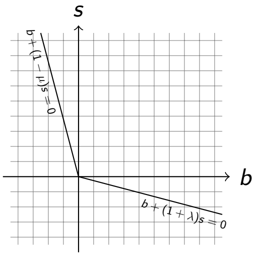

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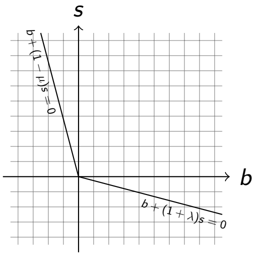

For a given initial position $(b, s) \in \mathcal{S}$, we denote the corresponding set of **admissible strategies** by $\mathcal{A}(b, s)$.

Problem Formulation

We consider the following **control problem**:

$$V(t, b, s) = \sup_{(L,M)\in\mathcal{A}(b,s)} \mathbb{E}_{t,b,s}\left[U_p\left(X_T\right)\right],$$

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where $U_p : \mathbb{R}_+ \to \mathbb{R}$ is a **utility function** of the form

$$U_p(x) = \begin{cases} \frac{1}{p}x^p, & \text{if } p < 1, p \neq 0, \\ \log(x), & \text{if } p = 0. \end{cases}$$

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This problem has been studied by many authors:

Magill/Constantinides (1976),
Davis/Norman (1990), Shreve/Soner (1994),
Akian/Séquier/Sulem (1995), Dai/Yi (2009), Bichuch (2012).

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Fix (L^*, M^*) and define

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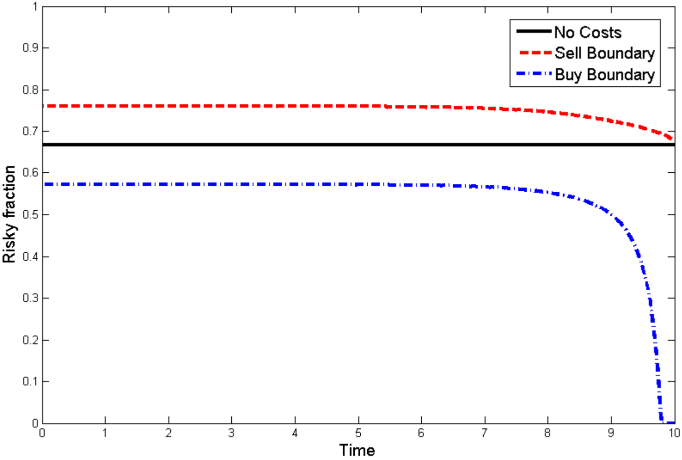
In total, this means h has to solve the **HJB equation**

$$0 = \max \left\{ \mathcal{L}^{nt} h(t, b, s), \mathcal{L}^{buy} h(t, b, s), \mathcal{L}^{sell} h(t, b, s) \right\}.$$

Numerical Example: Optimal Trading Regions



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- Uniqueness allows to treat the problem **numerically**.
- The existence of the optimal strategies is a **work in progress**.

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The idea behind Viscosity Solutions

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Viscosity Solutions: A Formal Definition

Assume that F is continuous and decreasing in its last argument. Let Ω be an open set and let w be continuous on Ω. Then

1. w is called a **viscosity subsolution** of $F = 0$, if for all $x_0 \in \Omega$ and all $\varphi \in C^2(\Omega)$ such that $w - \varphi$ has a local maximum at x_0, we have

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It turns out that the notion of viscosity solutions is the **perfect concept** for solutions of HJB-equations!

Application to our Transaction Cost Problem

Consider the **HJB equation**

$$0 = \min\{\mathcal{L}^{nt} h(t,b,s), \mathcal{L}^{buy} h(t,b,s), \mathcal{L}^{sell} h(t,b,s)\}, \qquad [0,T) \times \mathcal{S},$$

where the differential operators are given by

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Application to our Transaction Cost Problem

Consider the **HJB equation**

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Then the value function V is a **viscosity solution** with boundary conditons

$$\begin{aligned} V(t,b,s) &= U(0+), && \text{on } [0,T) \times \partial\mathcal{S}, \\ V(T,b,s) &= \begin{cases} U(b + (1-\mu)s), & \text{if } s > 0, \\ U(b + (1+\lambda)s), & \text{if } s \leq 0. \end{cases} && \text{on } \{T\} \times \mathcal{S}. \end{aligned}$$

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Therefore, if the HJB equation admits at most one viscosity solution, then the value function must also be a classical solution.

Outline

The Comparison Principle

In order to show uniqueness, one typically proves a **comparison principle**.

Comparison Principle

Let u be a viscosity subsolution and let v be a viscosity supersolution. Suppose that there exist constants $C > 0$ and $p \in (0, 1)$ such that

$$|u(t,x)| \leq C(1+|x|)^p, \qquad |v(t,x)| \leq C(1+|x|)^p.$$

If $u \leq v$ on the boundary, then $u \leq v$ on $[0, T] \times \overline{\mathcal{S}}$.

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Problem: The growth condition ensures that the value function does not grow to rapidly when x becomes large. If the value function satisfies the growth condition mainly depends on the choice of the utility function, i.e.

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The real problem is to control the value function **near the boundary**!

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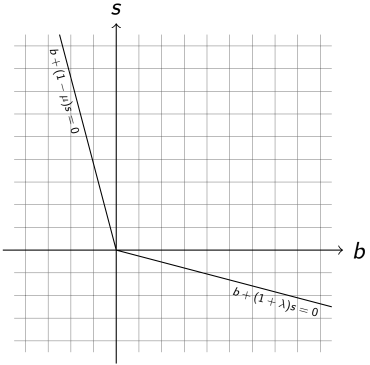
Suppose $u \leq v$ on $\partial\mathcal{S}$. We are interested in

$$\sup_{x \in \mathcal{S}} \{u(x) - v(x)\}.$$



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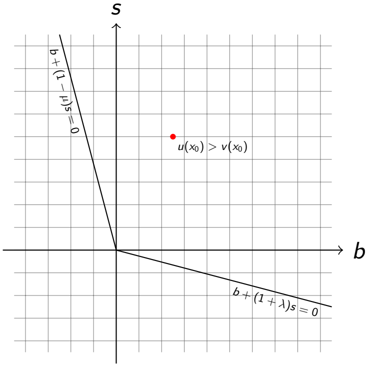

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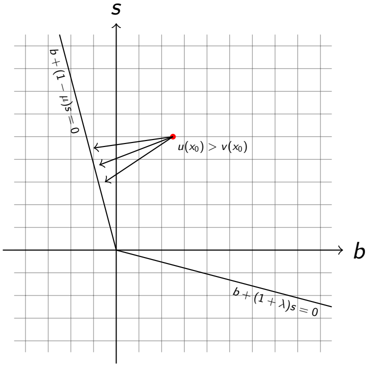

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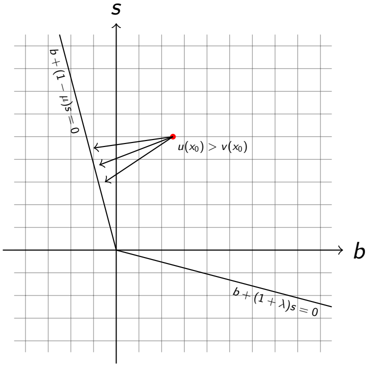
However, we cannot do this if we do not know how fast u and v grow.



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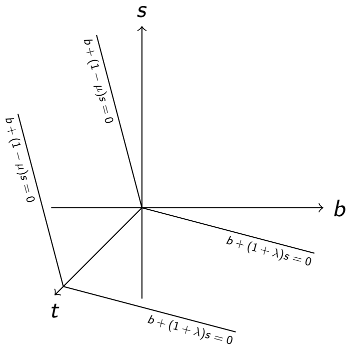

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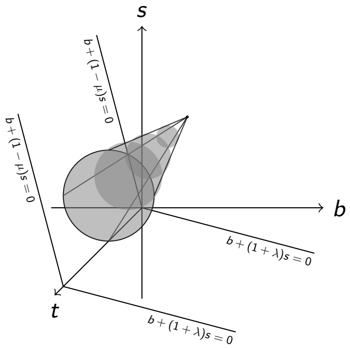

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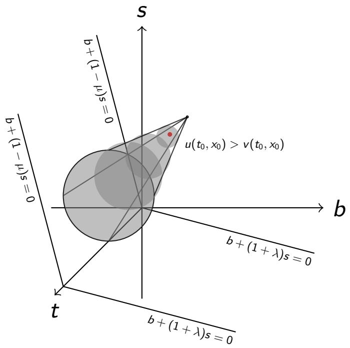

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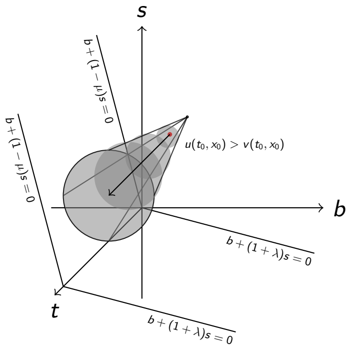

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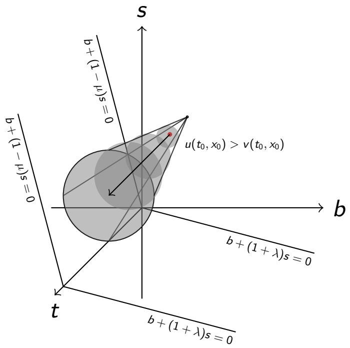

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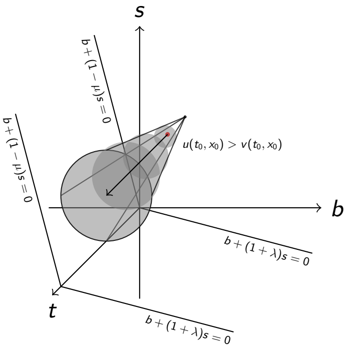
Since the cone is chosen arbitrarily, the comparison theorem holds everywhere.



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5. Next step: Show that the optimal controls exist.

Thank you for your attention!!!