Worst-Case Portfolio Optimization with Proportional Transaction Costs

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Abstract

We analyze optimal investment in crash-threatened markets for an investor facing proportional transaction costs. For every investment strategy, we identify a worst-case scenario in terms of the crashes with the objective of finding the strategy which performs best in its worst-case scenario.

Trading Strategies

Assume that the investor's wealth invested in bond and stock follows

$$dB_t = rB_t dt - (1+\lambda)dL_t + (1-\mu)dM_t,$$
$$dS_t = \alpha S_t dt + \sigma S_t dW_t + dL_t - dM_t.$$

L_t and M_t represent the cumulative purchases and sales of stock, respectively. L_t and M_t are assumed to be non-decreasing càdlàg processes. The constants λ and μ represent the proportional transaction costs incurred for buying and selling, respectively.

Market Crashes

A crash is modeled as a pair (τ, β_τ) consisting of a stopping time τ w.r.t. the filtration $\mathbb{F}^{W,L,M}$ and a crash level $\beta_\tau \in (0,\beta] \subset (0,1)$. When a crash occurs at time τ, then the stock price drops by a fraction of β_τ. Consequently,

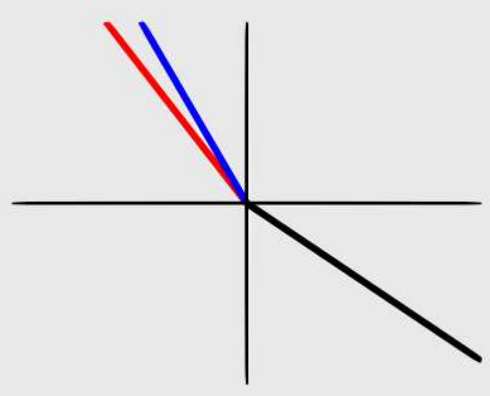
$$S_\tau = (1-\beta_\tau)S_{\tau-}.$$

We assume w.l.o.g. that $\beta_\tau = \beta$. The set of all crash times corresponding to a given trading strategy (L_t, M_t) is denoted by $\mathcal{B}(L,M)$.

Admissible Strategies

We assume that the investor can observe crashes and adjust her strategy once a crash has occurred. That is, she chooses a pre-crash strategy (L_t, M_t) to be used before the crash and a whole family of post-crash strategies $\{(L_t^\tau, M_t^\tau)\}_{\tau\in\mathcal{B}(L,M)}$ to be used after the crash. We denote the set of all pre-crash strategies starting at time t which for initial holdings (b,s) lead to non-negative wealth by $\mathcal{A}(t,b,s)$. The corresponding set of post-crash families is denoted by $\bar{\mathcal{A}}(L,M)$.

The non-negativity assumption leads to a constrained state space.



Problem Formulation

Our objective is to solve the following optimization problem:

$$V(t,b,s) = \sup_{\mathcal{A}(t,b,s),\bar{\mathcal{A}}(L,M)} \inf_{\mathcal{B}(L,M)} E_{t,b,s}\Big[U(X_T)\Big],$$

where X_T denotes the net wealth of the investor at terminal time $T > 0$ given by

$$X_T = \begin{cases} B_T + (1-\mu)S_T, & \text{if } S_T > 0,\\ B_T + (1+\lambda)S_T, & \text{if } S_T \le 0,\end{cases}$$

and where U is either power or log-utility.

The Bellman Principle

Denote by $\bar{\mathcal{V}}$ the value function in the absence of crashes. Then we can prove the following Bellman principle.

Theorem. *Let θ be a $[t,T]$-valued stopping time. Then*

$$\mathcal{V}(t,b,s) = \sup_{\mathcal{A}(t,b,s)} \inf_{\mathcal{B}(L,M)} E_{t,b,s}\Big[\mathcal{V}\big(\theta, B_{\theta-}, S_{\theta-}\big)\mathbf{1}_{\{\theta<\tau\}} \bar{\mathcal{V}}\big(\tau, B_{\tau-}, (1-\beta)S_{\tau-}\big)\mathbf{1}_{\{\theta\ge\tau\}}\Big].$$

This allows us to treat the optimality of pre-crash and post-crash strategies independently.

The Post-Crash Case

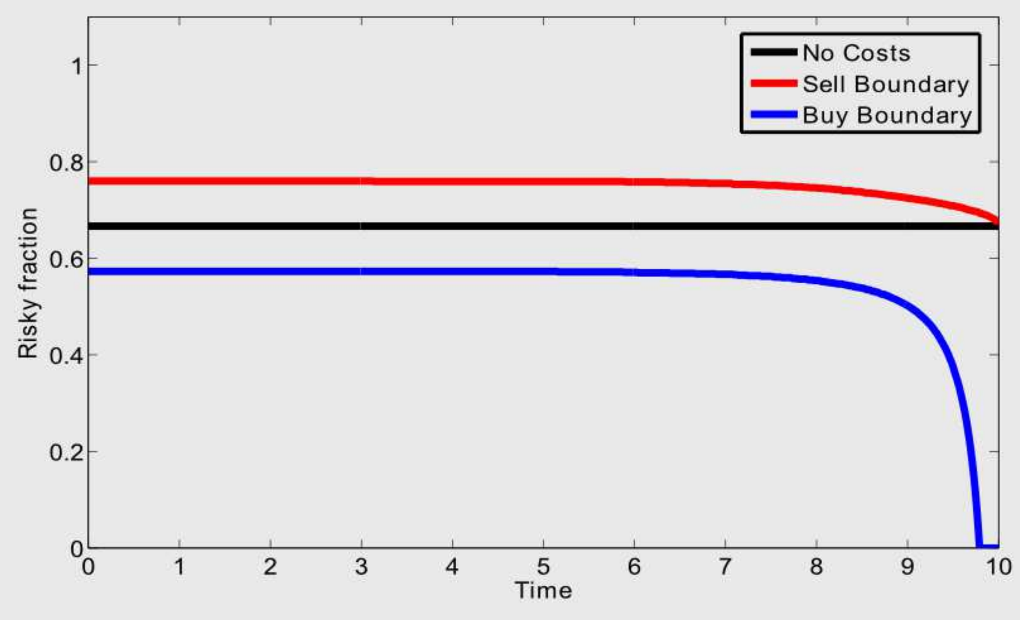
It is a well-known result, that $\bar{\mathcal{V}}$ is a viscosity solution of

$$0 = \min\Big\{\mathcal{L}^{nt}\bar{\mathcal{V}}(t,b,s), \mathcal{L}^{buy}\bar{\mathcal{V}}(t,b,s), \mathcal{L}^{sell}\bar{\mathcal{V}}(t,b,s)\Big\},$$

where the differential operators are given by

$$\mathcal{L}^{nt} = -\frac{\partial}{\partial t} - rb\frac{\partial}{\partial b} - \alpha s\frac{\partial}{\partial s} - \frac{1}{2}\sigma^2 s^2\frac{\partial^2}{\partial s^2},$$
$$\mathcal{L}^{buy} = (1+\lambda)\frac{\partial}{\partial b} - \frac{\partial}{\partial s},$$
$$\mathcal{L}^{sell} = -(1-\mu)\frac{\partial}{\partial b} + \frac{\partial}{\partial s}.$$

Uniqueness holds under growth conditions on $\bar{\mathcal{V}}$. Depending on which operator in the above equation vanishes determines the optimal action of the investor.



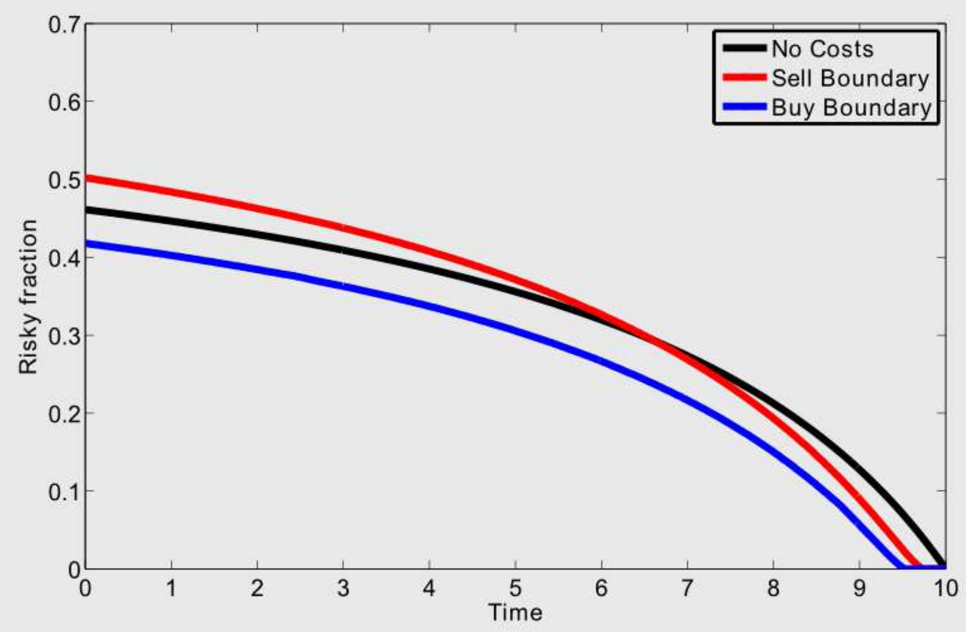
Parameters: $r = 0$, $\alpha = 0.096$, $\sigma = 0.4$, $\lambda = \mu = 0.01$, $p = 0.1$, $T = 10$.

The Pre-Crash Case

Similarly, $\mathcal{V}$ is a viscosity solution of

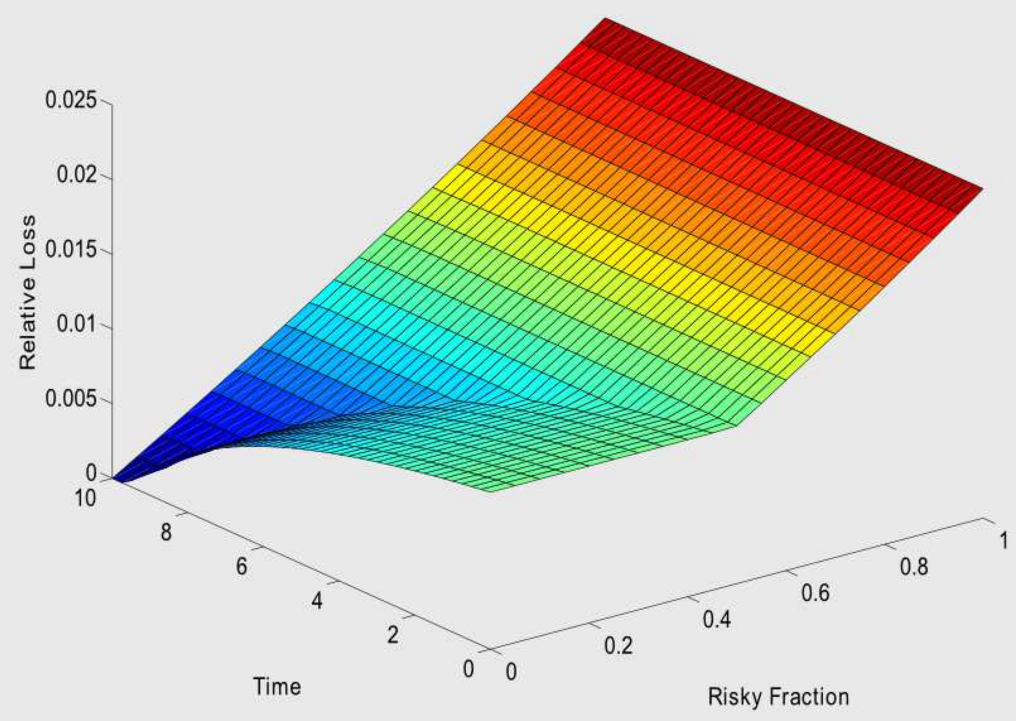
$$0 = \max\Big\{\mathcal{V}(t,b,s) - \bar{\mathcal{V}}(t,b,(1-\beta)s), \min\Big\{\mathcal{L}^{nt}\mathcal{V}(t,b,s), \mathcal{L}^{buy}\mathcal{V}(t,b,s), \mathcal{L}^{sell}\mathcal{V}(t,b,s)\Big\}\Big\}$$

Uniqueness holds under growth conditions on $\mathcal{V}$. The location of the trading regions for $\beta = 0.2$ is presented below.



Relative Loss of Utility

Clearly, an investor following the one-crash strategy loses expected utility if no crash occurs (compared to the no-crash strategy). However, the relative loss is small.



Extensions and Outlook

The results can easily be extended to an arbitrary but bounded number of crashes.

Future research may include:

- Extension of the results to different types of costs, e.g. constant costs or fixed costs.
- Worst-case optimization in the presence of liquidity constraints.
- Option pricing in the worst-case setting.

References

[1] Davis, M. H. A. and Norman, A. R. (1990). *Portfolio selection with transaction costs*, Math. Oper. Res. 15, 676–713.

[2] Shreve, S. E. and Soner, H. M. (1994). *Optimal investment and consumption with transaction costs*, Ann. Appl. Probab. 4, 609–692.

[3] Dai, M. and Yi, F.H. (2009). *Finite-horizon optimal investment with transaction costs: A parabolic double obstacle problem*, J. Differential Equations 246, 1445–1469.

[4] Korn, R. and Wilmott, P. (2002). *Optimal portfolios under the threat of a crash*, Int. J. Theor. Appl. Finance 5, 171–187.

[5] Korn, R. and Steffensen, M. (2007). *On worst-case portfolio optimization*, SIAM J. Control Optim. 46, 2013–2030.

[6] Seifried, F. T. (2010). *Optimal investment for worst-case crash scenarios: A martingale approach*, Math. Oper. Res. 35, 559–579.

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